Operationality in the Axiomatic Foundations of Relativity Theory

Attila Molnár   Gergely Székely

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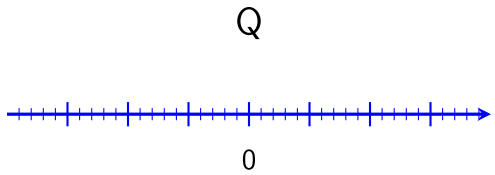
Hilbert's 6th problem

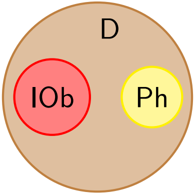
Our Classical approach

Results in the classical investigations

The classical approach: SpecRel

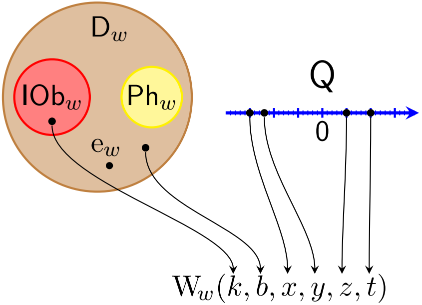


One sort for Mathematics: $\langle Q, +, \cdot, \leq \rangle$
AxEField: $\langle Q, +, \cdot, \leq \rangle$ is an euclidean field. (like $\mathbb{R}$.)



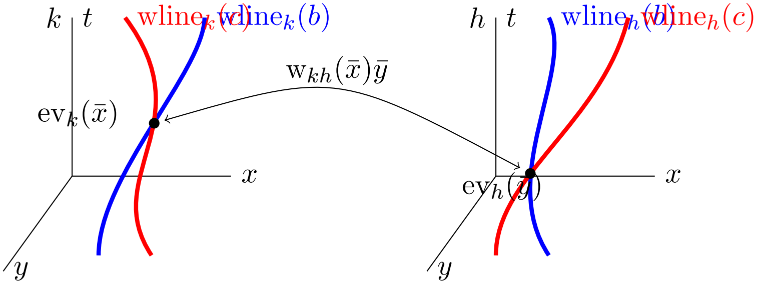
One sort for Physics:

Domain	D: the set of bodies
Photons	Ph: The set of photons.
Inertial Observers	IOb: The set of inertial observers.



World-view relation W: the relation of coordinatization.
"k coordinatizes b at the space-time location (x, y, z, t)."

Some definable concepts



$\mathrm{wline}_k(b)$:	The worldline of b according to k.
$\mathrm{ev}_k(\bar{x})$:	The event at $\bar{x}$ according to k.
$\mathrm{w}_{kh}(\bar{x}) = \bar{y}$:	Worldview-transformation between observers (inertial coordinate systems).

AxSelf: Every observer coordinatizes itself stationary in the origin:

$$(\forall k \in \mathrm{IOb})(\forall \bar{x} \in \mathrm{wline}_k(k)) \quad \bar{x}_s = \bar{0}.$$

AxSym: All observers use the same system of measurements:

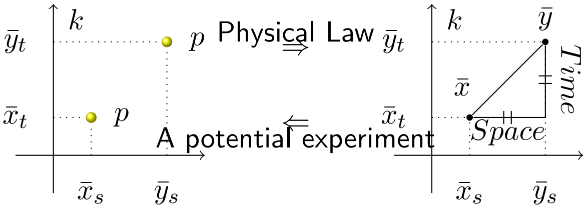
$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x}, \bar{x}', \bar{y}, \bar{y}')(|\bar{x}_t - \bar{y}_t| = 0 \wedge |\bar{x}'_t - \bar{y}'_t| = 0 \wedge$$
$$\wedge \mathrm{w}_{kh}(\bar{x}) = \bar{x}' \wedge \mathrm{w}_{kh}(\bar{y}) = \bar{y}') \rightarrow |\bar{x}_s - \bar{y}_s| = |\bar{x}'_s - \bar{y}'_s|.$$

Axioms

AxEv: The observers coordinatize the same events.

$$(\forall k, h \in \mathrm{IOb})(\forall \bar{x})(\exists \bar{y}) \quad \mathrm{w}_{kh}(\bar{x}) = \bar{y}.$$

AxPh: For any inertial observer, the speed of light is 1. Furthermore, it is possible to send out a light signal in any direction.



$$(\forall k \in \mathrm{IOb})(\forall \bar{x}, \bar{y})$$

$$[(\exists p \in \mathrm{Ph})\bar{x}, \bar{y} \in \mathrm{wline}_k(p)] \leftrightarrow \frac{|\bar{x}_s - \bar{y}_s|}{|\bar{x}_t - \bar{y}_t|} = 1$$

Extensionality

In SpecRel,

$$\text{“}\exists b\varphi\text{”}$$

means that

“There exists a **potential** body such that φ.”

Then what is the difference in SpecRel between

a **law** of Physics and a postulate saying that an **experiment** is executable?

Extensionality

Nothing.

That is the price of **extensionality**.

But what does this “extensionality” means?

Extensionality

Roughly speaking. . .

In an extensional framework, we treat Physics exactly in the same way as Mathematics:

Extensionality

A classical model of SpecRel is a **rigid totality**:

- Everything that could happen, happened already.
- There is nothing that **could have happened otherwise**.

But contrary to Mathematics, contrafactuals seem to be meaningful in Physics.

2+2=4, but it could have happened that 2+2=5.

vs.

b did not collided with e but it could have happened that they collided.

How can we keep the **rigor** of Mathematical Logic,
without the **rigidity** of it?

Extensionality

How can we keep the **rigor** of Mathematical Logic, without the **rigidity** of it?

There are intensional logics.

The modal solution:
Modal Logic:

> "a logic of some transformations between classical models"

Practically, a modal model is

- a set of classical models ("worlds"),
- and there is a relation on it, representing transformations between classical models.

KEP EGY KUSZA MODELLROL!!

Advantages of a Modal framework

The advantages of the modal approach:

- A natural axiomatization of Special Relativity: Difference between formalizations of
  - laws of Physics,
  - potential experiments.

Advantages of a Modal framework

The advantages of the modal approach:

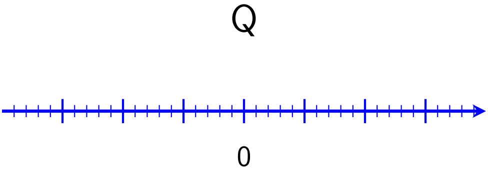
- A natural axiomatization of Special Relativity: Difference between formalizations of
  - laws of Physics,
  - potential experiments.
- New oppurtunities:
  SpecRel has an extension: SpecRelDyn. Its new primitive function:

  $$\mathrm{m}_k(b)\text{: ``the relativistic mass of } b \text{ according to } k\text{''}$$

  If we are in a modal framework, we can use ideas of **operationalism** and **define** this function using **kinematical** terms.

Modal (New) Approach: MSpecRel

Mathematics



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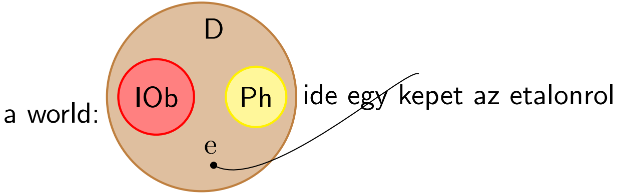
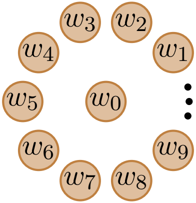


One **Modal** sort for Physics: Each world has a

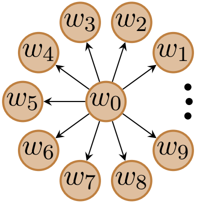
Domain	D: The domain of quantification, i.e., the set of **actual** or **existing** bodies in w.
Photons	Ph: The set of photons in w.
Inertial Observers	IOb: The set of inertial observers in w.
The standard-mass object	ε.

Worlds



Set of worlds $S = \{w_0, w_1, \dots\}$: Names of the extensional models $\langle D_w, \mathrm{Ph}_w, \mathrm{IOb}_w, \varepsilon_w \rangle$.

Worlds



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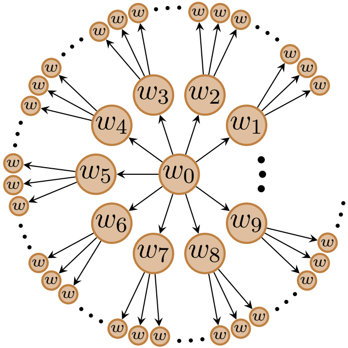
Worlds



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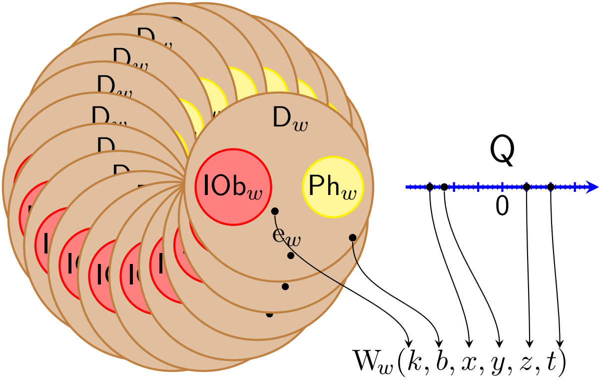


World-view relation W_w: the relation of coordinatization.
"k coordinatizes b at the space-time location (x, y, z, t)."

The new logical connectives: $\Diamond$, $\Box$.

$$w \models \Diamond\varphi[\sigma] \quad \stackrel{\text{def.}}{\Longleftrightarrow} \quad \text{there exists a } v\mathfrak{R}w\text{, such that } w \models \varphi[\sigma]$$
$$w \models \Box\varphi[\sigma] \quad \stackrel{\text{def.}}{\Longleftrightarrow} \quad \text{for all } v\mathfrak{R}w\text{, it is true that } w \models \varphi[\sigma]$$

Some possible interpretations:

- $\Diamond\varphi$: We could make an experiment according to which it will be the case that φ.
- $\Diamond\varphi$: This classical model can be transformed into another in which φ is true.
- $\Box\varphi$: φ is irrefutable by experiments.
- $\Box\varphi$: is invariant under some specific classical model-transformations.

Determining the alternative relation

to answer the question that	what is a potential experiment in MSpec
is the same as	to determine the relation R,
and this can be calibrated by	postulating a right selection of axioms containing $\Box$ and $\Diamond$.

Criteria to be a transformed version

- Minimal normal modal propositional logic **K** plus we assume reflexivity (**T**). (for simplicity)

$$\varphi \to \Diamond\varphi$$

- The standard mass is the same in the accessible worlds (it is a "rigid designator"), i.e.,

$$wRv \Rightarrow \mathrm{e}_w = \mathrm{e}_v$$

- The denotations of physical terms invariant under potential experiments,

$$b \neq c \to \Box b \neq c$$

$$b = c \to \Box b = c$$

- Existing observers remain existing observers.

$$(\forall k \in \mathrm{IOb})\Box(\underbrace{\exists b\, b = k}_{\mathrm{E}(b)} \wedge \mathrm{IOb}(k))$$

- Mathematics is invariant:

$$\begin{aligned} x + y = z &\leftrightarrow \Box x + y = z \\ x \cdot y = z &\leftrightarrow \Box x \cdot y = z \\ x \leq y &\leftrightarrow \Box x \leq y \end{aligned}$$

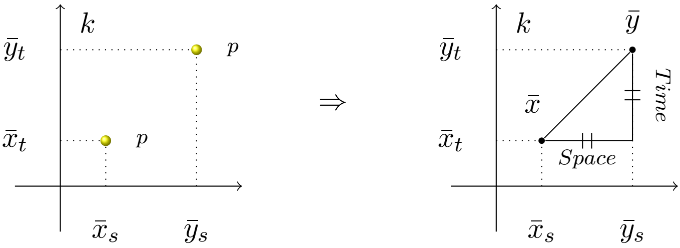
Sketch of a modal Kinematics

We have

- AxSelf: Every observer coordinatizes itself stationary in the origin:
- AxMSym: All observers use the same system of measurements:
- AxMEv: The observers coordinatize the same events.
- instead of AxPh...

AxPhObs:

Every observer sees the world-lines of photons as of slope 1.

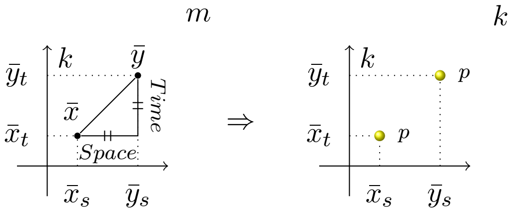


$$(\forall k \in IOb)(\forall \bar{x}, \bar{y})$$

$$(\exists p \in Ph)(\bar{x}, \bar{y} \in \mathrm{wline}_k(p)) \rightarrow \left(\frac{\mathrm{Space}(\bar{x}, \bar{y})}{\mathrm{Time}(\bar{x}, \bar{y})} = 1 \right) \quad (1)$$

Axiom of Signal Sending

AxPhExp: In every direction it is possible to send out a photon.



$$(\forall k \in \mathrm{IOb})(\forall \bar{x}, \bar{y} \in Q^4)$$

$$\left(\frac{\mathrm{Space}(\bar{x}, \bar{y})}{\mathrm{Time}(\bar{x}, \bar{y})} = 1 \rightarrow \Diamond(\exists p \in \mathrm{Ph})(\bar{x}, \bar{y} \in \mathrm{wline}_k(p)) \right) \quad (2)$$

$$\mathsf{MSpecRel} \stackrel{\text{def.}}{=} \{\mathsf{AxSelf}, \mathsf{AxMSym}, \mathsf{AxMEv}, \mathsf{AxPhObs}, \mathsf{AxPhExp}\}$$

Theorem

There are boole-invariant translations preserving truth between the classical and modal investigations:

$$\mathsf{SpecRel} \vdash \varphi \quad \Rightarrow \quad \mathsf{MSpecRel} \vdash \mathrm{Tr}^{+\Diamond}(\varphi)$$

$$\mathsf{MSpecRel} \vdash \varphi \quad \Rightarrow \quad \mathsf{SpecRel} \vdash \mathrm{Tr}^{-\Diamond}(\varphi)$$

But there is more:

We can define the mass explicitly in MSpecRel.

Definition of Mass

Definitions

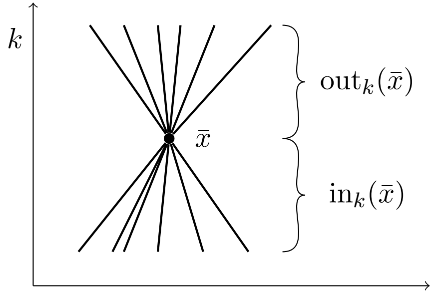
Definition (Inertial body)

Inertial bodies are the bodies with straight (line-subset) world-lines.

$$\mathrm{IB}(b) \stackrel{\text{def.}}{\Longleftrightarrow} (\exists k \in \mathrm{IOb})(\forall \bar{x}, \bar{y}, \bar{z} \in \mathrm{wline}_k(b))$$

$$(\bar{x}_t \leq \bar{y}_t \leq \bar{z}_t \rightarrow |\bar{x} - \bar{y}| + |\bar{y} - \bar{z}| = |\bar{x} - \bar{z}|)$$

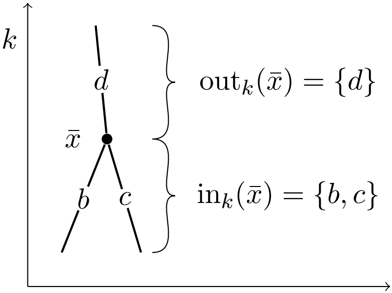
Incoming and Outgoing



Definition (Incoming and Outgoing Inertial Bodies)

$$\mathrm{in}_k(\bar{x}) \stackrel{\text{def.}}{=} \{b \in \mathrm{IB} : b \in \mathrm{ev}_k(\bar{x}) \wedge (\forall \bar{y} \in \mathrm{wline}_k(b))\, \bar{y}_t < \bar{x}_t \vee \bar{y} = \bar{x}\}$$
$$\mathrm{out}_k(\bar{x}) \stackrel{\text{def.}}{=} \{b \in \mathrm{IB} : b \in \mathrm{ev}_k(\bar{x}) \wedge (\forall \bar{y} \in \mathrm{wline}_k(b))\, \bar{y}_t > \bar{x}_t \vee \bar{y} = \bar{x}\}$$

Collision



Definition (Collision)

A (2-body) inelastic collision happens when two bodies comes and one body left.

$$\mathrm{inecoll}_{k,\bar{x}}(b, c : d) \overset{\text{def.}}{\iff} \begin{pmatrix} b, c, d \in \mathrm{E} \\ b \neq c \\ \mathrm{in}_k(\bar{x}) = \{b, c\} \\ \mathrm{out}_k(\bar{x}) = \{d\} \end{pmatrix}$$

Abbreviations

The "missing variables" are bounded by existential quantification:

$$\begin{aligned}
\mathrm{inecoll}_{k,\bar{x}}(b,c) &\overset{\text{def.}}{\iff} (\exists d \in \mathrm{IB})\mathrm{inecoll}_{k,\bar{x}}(b,c:d)\\
\mathrm{inecoll}_{k}(b,c) &\overset{\text{def.}}{\iff} (\exists \bar{x})\mathrm{inecoll}_{k,\bar{x}}(b,c)\\
\mathrm{inecoll}(b,c) &\overset{\text{def.}}{\iff} (\exists k \in \mathrm{IOb})\mathrm{inecoll}_{k}(b,c)
\end{aligned}$$

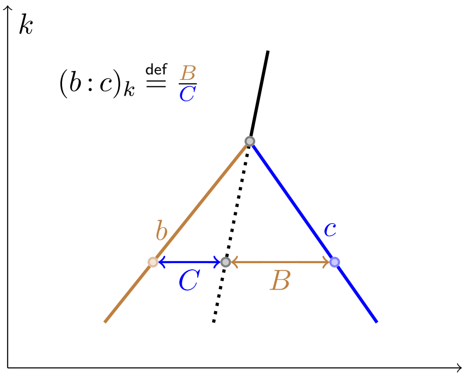
Definition (Covering Line)

$$\overline{\mathrm{wline}}_k(d) \stackrel{\text{def.}}{=} \left\{ \bar{z} \,:\, (\forall \bar{x}, \bar{y} \in \mathrm{wline}_k(d)) \begin{pmatrix} |\bar{x}-\bar{y}| + |\bar{y}-\bar{z}| = |\bar{x}-\bar{z}| \vee \\ |\bar{x}-\bar{z}| + |\bar{z}-\bar{y}| = |\bar{x}-\bar{y}| \vee \\ |\bar{z}-\bar{x}| + |\bar{x}-\bar{y}| = |\bar{z}-\bar{y}| \end{pmatrix} \right\}$$

$$\overline{\mathrm{wline}}_k(d,t) = \bar{s} \stackrel{\text{def.}}{\Longleftrightarrow} \langle \bar{s}, t \rangle \in \overline{\mathrm{wline}}_k(d)$$

$$\mathrm{wline}_k(d,t) = \bar{s} \stackrel{\text{def.}}{\Longleftrightarrow} \langle \bar{s}, t \rangle \in \mathrm{wline}_k(d)$$

Collision Ratio



Definition (Ratio of Collision)

$$(b:c)_k = r \stackrel{\text{def.}}{\iff} \text{inecoll}(b,c) \land$$
$$\land (\exists t < \text{locinecoll}_k(b,c))\, r = \frac{|\text{loc}_k(c,t) - \overline{\text{wline}}_k(d,t)|}{|\text{loc}_k(b,t) - \overline{\text{wline}}_k(d,t)|}$$

Collision Experiment

Definition (Colliding Experiments)

In a world w a body c is **collidable** to b according to k iff

- b is an existing inertial body in w
- k is an existing inertial observer in w,
- there is an alternative world w' where these are still existing, inertial,
- in w' there is an inertial body c colliding with b.

We call such a $\langle w, w', k, b, c\rangle$ tuple a **colliding (thought-)experiment**.

But we are interested in those collidings,
where the b has the same speed/world-line as before.

Relevant Collision Experiment

Definition

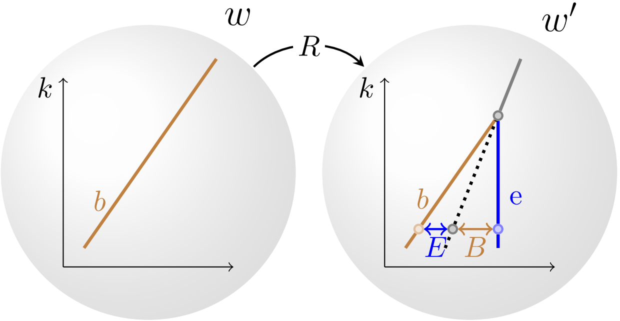
In a world w a body c is **relevantly** collidable to b according to k

iff

the world-line of b before the collision is the same in both world according to k, that is,

$$(\forall t \leq \mathrm{locinecoll}_k(b:c)_t)\mathrm{wline}_k(b,t)_w = \mathrm{wline}_k(b,t)_{w'}.$$

Direct Measurement



$$\mathrm{m}^-_k(b) \overset{\text{def}}{=} \frac{B}{E} \qquad (b:\mathrm{e})_k = \frac{B}{E}$$

Definition (Direct Measurement)

A colliding experiment $\langle w, w', k, b, c\rangle$ is a **direct measurement** iff it is relevant, c is the etalon and it is stationary according to k.

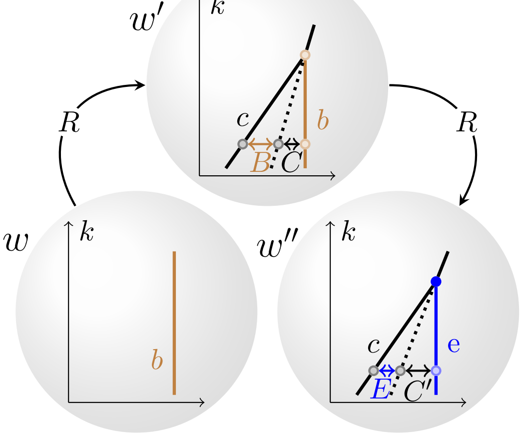
Indirect Measurement

Definition (Indirect Measurement)

A colliding experiment $\langle w, w', k, b, c\rangle$ is an **indirect measurement** iff

- it is relevant,
- b is stationary,
- there exists a direct measurement $\langle w', w'', k, c, \varepsilon\rangle$.

$$(b:c)_k = \frac{B}{C}$$



$$\mathrm{m}^0_k(b) \stackrel{\text{def}}{=} \frac{B}{C}\cdot\frac{C'}{E} \qquad (c:\mathrm{e})_k = \frac{C'}{E}$$

AxCollRel: Every colliding experiment is relevant.

$$
\begin{array}{l}(\forall k \in \mathrm{IOb})\\(\forall b \in \mathrm{IB})\\(\forall \bar{x}, \bar{y})\end{array}
\left(\left(\begin{array}{c}\bar{y}_t \leq \bar{x}_t\\ \mathrm{W}(k, b, \bar{y})\end{array}\right) \rightarrow \Box((\exists c \in \mathrm{IB})\mathrm{inecoll}_{k,\bar{x}}(b, c) \rightarrow \mathrm{W}(k, b, \bar{y}))\right)
$$

AxDir: According to any inertial observer, every relatively moving inertial body is collidable uniquely with the stationary etalon.

$$
\begin{array}{l}(\forall k \in \mathrm{IOb})\\(\forall b \in \mathrm{IB})\\(\exists r)\end{array}
\left(\begin{array}{l}\Diamond(\mathrm{v}_k(\varepsilon) = 0 \wedge (b : \varepsilon)_k = r)\\ \Box(\mathrm{v}_k(\varepsilon) = 0 \rightarrow (b : \varepsilon)_k = r)\end{array}\right)
$$

AxIndir: For every inertial observer, every resting body is involved in an indirect measurement, and the result of it is unique, that is, invariant under the choice of the transmitting body.

$$
\begin{array}{l}(\forall k \in \mathrm{IOb})\\(\forall b \in \mathrm{IB})\end{array}
\mathrm{v}_k(b) = 0 \rightarrow (\exists! r)\left(\begin{array}{l}\Diamond(\exists c \in \mathrm{IB}) r = (b : c)_k \cdot \mathrm{m}_k^-(c)\\ \Box(\forall c' \in \mathrm{IB}) r = (b : c')_k \cdot \mathrm{m}_k^-(c')\end{array}\right)
$$

Relativistic Mass

Definition ("Moving Mass")

If AxDir is assumed, then mass of relatively moving body is the collision ratio granted by the direct measurement(s):

$$\mathrm{m}_k^-(b) = r \overset{\text{def.}}{\iff} \Diamond(\mathrm{v}_k(\varepsilon) = 0 \wedge (b:\varepsilon)_k = r)$$

Definition (Rest Mass)

If AxDir and AxIndir are assumed, then the rest mass of a stationary body is product of the collision ratios granted by the indirect measurement(s):

$$\mathrm{m}_k^0(b) = r \overset{\text{def.}}{\iff} \mathrm{v}_k(b) = 0 \wedge \Diamond(\exists c \in \mathrm{IB})(r = (b:c)_k \cdot \mathrm{m}_k^-(c))$$

In case of the rest mass, we can omit naming the observer:

$$\mathrm{m}_0(b) = r \overset{\text{def.}}{\iff} (\exists k \in \mathrm{IOb})\mathrm{m}_k^0(b) = r.$$

Definition (Relativistic Mass)

$$\mathrm{m}_k(b) \overset{\text{def.}}{=} \begin{cases} \mathrm{m}_k^-(b), & \text{if } \mathrm{v}_k(b) \neq 0, \\ \mathrm{m}_k^0(b), & \text{if } \mathrm{v}_k(b) = 0. \end{cases}$$

Example: Rest mass of the etalon

$$\mathrm{m}_0(\varepsilon) = (\varepsilon : c)_k \cdot \mathrm{m}_k^-(c) = (\varepsilon : c)_k \cdot (c : \varepsilon)_k = 1$$

In one of the colliding bodies is at rest according to an observer, then it is true that the collision ratio is the ratio of masses:

$$(b:c)_k = \frac{1}{(c:b)_k} = \frac{\mathrm{m}_k^-(b)}{(c:b)_k \cdot \mathrm{m}_k^-(b)} = \frac{\mathrm{m}_k^-(b)}{\mathrm{m}_k^0(c)} = \frac{\mathrm{m}_k(b)}{\mathrm{m}_k(c)}.$$

but this is contingent in general.

AxCenter: The ratio of a collision is the ratio of the masses of the colliding bodies.

$$\begin{matrix}(\forall k \in \mathrm{IOb})\\ (\forall b, c \in \mathrm{IB})\end{matrix} \left(\begin{pmatrix} \mathrm{v}_k(b) \neq 0 \\ \mathrm{v}_k(c) \neq 0 \\ \mathrm{inecoll}(b,c) \end{pmatrix} \rightarrow (b : c)_k = \frac{\mathrm{m}_k(b)}{\mathrm{m}_k(c)} \right)$$

This axiom builds the bridge between the worlds in the case of collisions: From now on, every collision will governed by possible collisions with the etalon.

But sometimes, one standard-mass is not enough.

Sometimes we need every observer to have its own standard-mass.
These will be the **standard-mass equivalents**.
Properties:

- If a standardmass-equivalents collides with the standard mass, then a median observer coordinatizes the resulting body of the collision at rest. (d)
- If a standardmass-equivalents collides with another equivalent, then a median observer coordinatizes the resulting body of the collision at rest.

Etalon-equivalence

Definition (Median Observer)

An observer m is median of the collision consisting body b and c iff the velocity of b and c are opposite for m.

$$\mathrm{MedianOb}_{b,c}(m) \overset{\text{def.}}{\iff} \bar{\mathrm{v}}_m(b) + \bar{\mathrm{v}}_m(c) = \bar{0}$$

Definition (Symmetric Collision)

A collision is symmetric iff there is a median observer of it coordinatizing the resulting body at rest.

$$\mathrm{SymColl}(b,c) \overset{\text{def.}}{\iff} \mathrm{inecoll}(b,c) \wedge (\exists m \in \mathrm{MedianOb}_{b,c})(b:c)_m = 1$$

Symmetric Collision Theorem

Theorem (Symmetric Collision Theorem)

The symmetric collisions have the collision ratio of $\sqrt{1-v^2}$ *according to the co-moving observers.*

$$\mathsf{MSpecRel} \vdash \begin{array}{l}(\forall b, c \in \mathrm{IB})\\(\forall k, l \in \mathrm{IOb})\end{array} \left(\begin{array}{l}\mathrm{SymColl}(b,c)\\ \mathrm{v}_k(b) = 0\\ \mathrm{v}_l(c) = 0\end{array}\right) \to (b:c)_k = \sqrt{1 - \mathrm{v}_l(k)^2}$$

Etalon-Equivalence

Definition (Etalon-equivalence)

A body b is an equivalent of the etalon of k iff b is co-moving with k and whenever it collides with the etalon, it collides with it symmetrically.

$$\mathrm{Et}_k(b) \overset{\text{def.}}{\iff} \mathrm{v}_k(b) = 0 \land \Box(\mathrm{inecoll}(b, \varepsilon) \to \mathrm{SymColl}(b, \varepsilon))$$

Pseudo-Direct Measurements

AxPDirM: For every observer k and every non-stationary inertial body b according to k, there is an alternative world in which b collides with a resting etalon-equivalent e_k of k, and this collision has a median observer.

$$\begin{matrix}(\forall k \in \mathrm{IOb}) \\ (\forall b \in \mathrm{IB})\end{matrix} \; \Diamond(\exists \mathrm{e}_k \in \mathrm{Et}_k)\mathrm{inecoll}(b, \mathrm{e}_k) \wedge (\exists m)\mathrm{MedianOb}_{b,\mathrm{e}_k}(m)$$

Modal Dynamics

$$\mathsf{MSpecRelDyn} \stackrel{\text{def.}}{=} \mathsf{MSpecRel} \cup \left\{ \begin{array}{l} \mathsf{AxCollRel} \\ \mathsf{AxDir} \\ \mathsf{AxIndir} \\ \mathsf{AxPDirM} \\ \mathsf{AxCenter} \end{array} \right\}$$

Sketch of the Mass Growth Theorem

Theorem (Collision of Equivalents)

MSpecRelDyn $\vdash$ *The equivalents of etalon collides each other symmetrically.*

Theorem (Mass of the equivalents)

MSpecRelDyn $\vdash$ *The rest mass of the equivalents of the etalon are* 1.

Theorem (Etalon-equivalence)

MSpecRelDyn $\vdash$ *The etalon can substituted by its co-moving equivalents.*

Sketch of the Mass Increase Theorem

$$
\begin{aligned}
\mathrm{m}_0(b) &= \mathrm{m}_h^0(b) && (w)\\
&= (b:c)_h \cdot \mathrm{m}_h^-(c) && (w^*)\\
&= (b:\mathrm{e}_k)_h \cdot \mathrm{m}_h^-(\mathrm{e}_k) && (w')\\
&= (b:\mathrm{e}_k)_k \cdot (1-v^2) \cdot \mathrm{m}_h^-(\mathrm{e}_k) && (w')\\
&= (b:\mathrm{e}_k)_k \cdot (1-v^2) \cdot \frac{1}{\sqrt{1-v^2}} && (w')\\
&= (b:\mathrm{e}_k)_k \cdot \sqrt{1-v^2} && (w')\\
&= (b:\varepsilon)_k \cdot \sqrt{1-v^2} && (w')\\
&= \mathrm{m}_k(b) \cdot \sqrt{1-v^2} && (w)
\end{aligned}
$$