Quantum Probabilities and Non Distributive Lattices

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Foundations Of Physics-2013/Munich/31.07.2013

Outline

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Abstract

We study the origin of quantum probabilities as arising from non-boolean propositional-operational structures. We apply the method developed by Cox to non distributive lattices and deduce non-kolmogorvian probability measures of quantum mechanics. We discuss the relationship between a group theoretical notion of object and the problems posed by von Neumann regarding the development of a quantum probability theory.

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Regularities

Given a physical theory:

- We concentrate on *events* or *states of affairs*. Given a certain state of affairs, we ask for how likely is that another state of affairs occurs. How can we measure this?
- How are events of a given theory structured? Are they structured in some way? Is there a link between the event structure and probability theory?
- The existence of well defined events is a necessary condition in order to have a scientific theory. This is independent of the position that we adopt, realism, empiricism, etc.

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Different kinds of regularities

Several possibilities:

- We can predict with absolute certainty (example: classical mechanics, determinism). *Absolute regularity*.
- We can only compute probabilities: given two equivalent preparations, we may have different effects, but with probabilities cogently defined (example: quantum mechanics). *Statistical regularity*.
- There is no regularity at all. Several reasons: ontological, impossibility of defining equivalent preparations, ect. Even in this case, we can (sometimes) define probabilities (but we will not discuss this here).

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Projective Measurements

Projective Measures

- Let $M : B(\mathcal{R}) \to \mathcal{P}(\mathcal{H})$
- $M(0) = 0$
- $M(\mathcal{R}) = \mathbf{1}$
- $M(\cup_j(B_j)) = \sum_j M(B_j)$ for any disjoint denumerable family B_j.
- $M(B^c) = \mathbf{1} - M(B) = (M(B))^{\perp}$

Connection with observables

M is a projective measure. It defines an observable (because of the spectral resolution theorem). $\mathcal{P}(\mathcal{H})$ is an event algebra (YES-NO tests).

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How Are Questions Structured?

The set of events of important examples of scientific theories can be endowed with well defined structures:

- Events in CM (subsets of phase space) form a boolean algebra (a *classical logic*).
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Definitions

Partially ordered set (poset)

- For all $x, y \in X$, $x < y$ and $y < x$, we have $x = y$
- For all $x, y, z \in X$, if $x < y$ y $y < z$, then $x < z$

Lattice

A lattice is a Poset for which any pair of elements x and y has an infimum $x \wedge y$ and a supremum $x \vee y$. If the infimum and the supremum exist for arbitrary collections, we have a *complete lattice*.

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- $\neg(\neg(a)) = a$
- $a \leq b \longrightarrow \neg b \leq \neg a$
- For all a, $(a \vee \neg a)$ and $(a \wedge \neg a)$ exist and they satisfy $a \vee \neg a = \mathbf{1}$ y $a \wedge \neg a = \mathbf{0}$.

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- $x \vee x = x$, $x \wedge x = x$ (idempotence)
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- $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associativity)
- $x \vee (x \wedge y) = x \wedge (x \vee y) = x$ (absorption)
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity 1)
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Boolean algebra

An orthocomplemented distributive lattice is a *Boolean algebra*.

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Kolmogorov

- $\mu : \Sigma \to [0, 1]$
- satisfying $\mu(\emptyset) = 0$
- $\mu(A^c) = 1 - \mu(A)$, where $(\ldots)^c$ means set theoretical complement and
- for any denumerable and pairwise disjoint collection $\{A_i\}_{i \in I}$ $\mu(\bigcup_{i \in I} A_i) = \sum_i \mu(A_i)$

Booleanity And Sum Rule

Σ is a boolean algebra and it satisfies $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ (*inclusion-exclusion principle*). We also have: $\mu(A \cup B) \leq \mu(A) + \mu(B)$

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R. T. Cox (1946)

- Cox starts with the boolean algebra of propositions (the propositional calculus of classical logic).
- Studying the symmetries of the boolean algebra, he deduces the general form of the possible measures.
- He deduces Kolmogorovian probability theory with this method (sum rule, inclusion-exclusion principle, product rule, Bayes formulae, etc.).

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Kolmogorov is not the end of the story...

Non-kolmogorovian probability

- Define a map $s : \mathcal{P}(\mathcal{H}) \to [0; 1]$ such that
- $s(\mathbf{0}) = 0$ ($\mathbf{0}$ is the nule subspace)
- $s(P^{\perp}) = 1 - s(P)$
- for any denumerable and pairwise orthogonal collection of projections P_j, $s(\sum_j P_j) = \sum_j s(P_j)$.

Quantum Probability

s is defined on a non-boolean algebra: $\mathcal{P}(\mathcal{H})$ is an orthomodular lattice. Gleason: $\Longrightarrow \forall s\, \exists \rho$ such that $s(P) = \mathrm{tr}(\rho P)$. $s(a \vee b) \geq s(a) + s(b)$.

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Questions

- What happens if the Cox's method is applied to non-boolean algebras?
- Does the underlying logical structure of a given theory determines its probability theory? What happens with information?
- What about ontology? Can we provide an ontological framework?

[Goyal-Knuth] P. Goyal and K. Knuth, Symmetry, **3**, 171-206 (2011) doi:10.3390/sym3020171.
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von Neumann:

"In order to have probability all you need is a concept of all angles, I mean, other than 90. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leaves all angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logic are fixed under some isomorphic transformation, all of probability theory is already fixed... This means however, that one has a formal mechanism in which, logics and probability theory arise simultaneously and are derived simultaneously."

"It was simultaneous emergence and mutual determination of probability and logic what von Neumann found intriguing and not at all well understood. He very much wanted to have a detailed axiomatic study of this phenomenon because he hoped that it would shed "... a great deal of new light on logics and probability alter the whole formal structure of logics considerably, if one succeeds in deriving this system from first principles, in other words from a suitable set of axioms." He emphasized –and this was his last thought in his address– that it was an entirely open problem whether/how such an axiomatic derivation can be carried out."

[M. Redei] M. Rédei, *The Mathematical Intelligencer*, **21**, (1999) 7-12.

Outline

General Method

We proceed as follows:

- Let $\mathcal{L}$ be an orthomodular lattice.
- We assume that $\mathcal{L}$ represents the propositional structure of a given system.
- We assume that there is a given state of affairs determined by the preparation of the system.
- We define a function $s : \mathcal{L} \longrightarrow \mathbb{R}$ such that $s(a) \geq 0 \forall a \in \mathcal{L}$ and it is order preserving ($a \leq b \longrightarrow s(a) \leq s(b)$).
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General Method

We proceed as follows:

- Let $\mathcal{L}$ be an orthomodular lattice.
- We assume that $\mathcal{L}$ represents the propositional structure of a given system.
- We assume that there is a given state of affairs determined by the preparation of the system.
- We define a function $s : \mathcal{L} \longrightarrow \mathbb{R}$ such that $s(a) \geq 0 \forall a \in \mathcal{L}$ and it is order preserving ($a \leq b \longrightarrow s(a) \leq s(b)$).
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It is possible to show that:

- $s(\bigvee\{a_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} s(a_i)$
- $s(\neg a) = 1 - s(a)$
- $s(\mathbf{0}) = 0$

The probability theory defined by these equations is non classical in the general case. If $\mathcal{L}$ is not Boolean, it may happen that $(a \wedge b) \vee (a \wedge \neg b) < a$.
and then, (using $(a \wedge \neg b) \perp (a \wedge b)$),
$s((a \wedge \neg b) \vee (a \wedge b)) = s(a \wedge \neg b) + s(a \wedge b) \leq s(a)$. But any Kolmogorovian probability satisfies $s(a) = s(a \wedge b) + s(a \wedge \neg b)$.

General Method

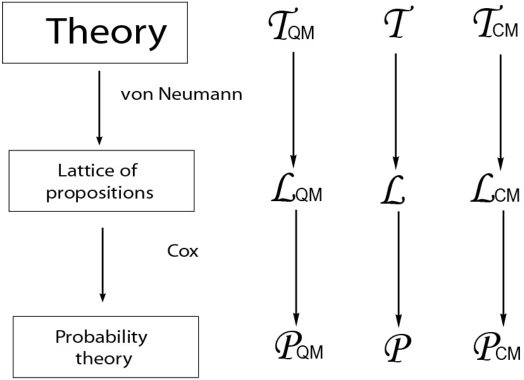

Figure : Schematic diagram of the method.

Outline

The program posed by von Neumann seems to point in the direction of a special kind of objects...

- Automorphisms of the logic seem to play a key role in the program proposed by von Neumann.
- In concrete implementations of quantum observables, symmetry transformations must preserve probability.
- Thus, we have objects such that its propositional latice is non-boolean, and that probability must remain invariant under these automorphisms.
- Which ontology is compatible with objects of this kind? We look for alternatives.

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Group Theoretical Quantum Ontology

- A quantum object will be an entity having objective properties, but at the same time, each objective property will have associated a symmetry transformation (represented by the action of a group).
- The action of this group will determine in turn a set of non-objective properties.
- To each objective property we can associate an operator. Each element of the orbit defined by this operator will be a phase of the object.

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In this way...

- A phase transformation is generated by the operator associated to an objective property. This transformation is induced by the objective property.
- The action of the groups associated to each objective property will leave invariant the set of objective properties.
- Thus, a physical system is a multifaceted structure characterized by a set of invariant objective properties (a kind of *dice* with (possibly) superposed faces).

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Outline

- According to von Neumann, a framework in which logic and probability theory are derived simultaneously is in order. In this formulation the automorphisms of the logic will play a key role.
- We have presented a formal framework in which probability theory is developed out of the algebraic properties of the underlying logic of the theory. Thus, we have presented a possible solution to the problem posed by von Neumann.

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- We have presented a formal framework in which probability theory is developed out of the algebraic properties of the underlying logic of the theory. Thus, we have presented a possible solution to the problem posed by von Neumann.

- Given that automorphisms of the logic are the elements which allow for the definition of objectivity in the GTQO ontology.
- Thus, the notion of GTQO, could provide an ontological framework for the program delineated by von Neumann. This is what we want to explore.
- This ontology $+$ the generalization of Cox's method to non-distributive lattices, could provide a complete realization of the program delineated by von Neumann.

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References

- [R. T. Cox] Cox, R.T. The Algebra of Probable Inference; The Johns Hopkins Press: Baltimore, MD, USA, (1961).
- [QL] G. Birkhoff and J. von Neumann, Annals Math.**37** (1936) 823-843.
- [Probability and QL] M. Rédei, *The Mathematical Intelligencer*, **21**, (1999) 7-12.
- [Application of Cox 1] P. Goyal and K. Knuth, Symmetry, **3**, 171-206 (2011) doi:10.3390/sym3020171.
- [Application of Cox 2] F. Holik, A. Plastino and M. Sáenz, http://arxiv.org/abs/1211.4952.
- [GTQO] G. Catren, “On Classical and Quantum Objectivity, *Foundations of Physics*, **38**, 470-487.