Quantum Probabilities and Non Distributive Lattices

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Abstract

General Framework
  Lattices
  Posets and lattices
  Boolean algebra

Quantum probabilities
  Kolmogorov
  R. T. Cox
  Quantum Probabilities
  The von Neumann program

General Scheme
  General Method

Group theoretical quantum objects

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Conclusions
We study the origin of quantum probabilities as arising from non-boolean propositional-operational structures. We apply the method developed by Cox to non distributive lattices and deduce non-kolmogorvian probability measures of quantum mechanics. We discuss the relationship between a group theoretical notion of object and the problems posed by von Neumann regarding the development of a quantum probability theory.
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Regularities

Given a physical theory:

- We concentrate on *events* or *states of affairs*. Given a certain state of affairs, we ask for how likely is that another state of affairs occurs. How can we measure this?
- How are events of a given theory structured? Are they structured in some way? Is there a link between the event structure and probability theory?
- The existence of well defined events is a necessary condition in order to have a scientific theory. This is independent of the position that we adopt, realism, empiricism, etc.
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Different kinds of regularities

Several possibilities:

- We can predict with absolute certainty (example: classical mechanics, determinism). *Absolute regularity.*

- We can only compute probabilities: given two equivalent preparations, we may have different effects, but with probabilities cogently defined (example: quantum mechanics). *Statistical regularity.*

- There is no regularity at all. Several reasons: ontological, impossibility of defining equivalent preparations, etc. Even in this case, we can (sometimes) define probabilities (but we will not discuss this here).
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Projective Measurements

Projective Measures

- Let $M : B(\mathcal{R}) \rightarrow \mathcal{P}(\mathcal{H})$
- $M(0) = 0$
- $M(\mathcal{R}) = 1$
- $M(\bigcup_j B_j) = \sum_j M(B_j)$ for any disjoint denumerable family $B_j$.
- $M(B^c) = 1 - M(B) = (M(B))^\perp$

Connection with observables

$M$ is a projective measure. It defines an observable (because of the spectral resolution theorem). $\mathcal{P}(\mathcal{H})$ is an event algebra (YES-NO tests).
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How Are Questions Structured?

The set of events of important examples of scientific theories can be endowed with well defined structures:

- Events in CM (subsets of phase space) form a boolean algebra (a *classical logic*).
- Events in QM (closed subspaces of Hilbert space) form an orthomodular lattice (a *quantum logic*).
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- Events in QM (closed subspaces of Hilbert space) form an orthomodular lattice (a quantum logic).
Posets and lattices

Definitions

**Partially ordered set (poset)**

- For all \( x, y \in X \), \( x < y \) and \( y < x \), we have \( x = y \)
- For all \( x, y, z \in X \), if \( x < y \) \( y < z \), then \( x < z \)

**Lattice**

A lattice is a Poset for which any pair of elements \( x \) and \( y \) has an infimum \( x \land y \) and a supremum \( x \lor y \). If the infimum and the supremum exist for arbitrary collections, we have a complete lattice.
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**Lattice**

A lattice is a Poset for which any pair of elements $x$ and $y$ has an infimum $x \wedge y$ and a supremum $x \vee y$. If the infimum and the supremum exist for arbitrary collections, we have a *complete lattice*. 
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Definitions

**Orthocomplemented Poset (orthoposet)**

- \( \neg(\neg(a)) = a \)
- \( a \leq b \rightarrow \neg b \leq \neg a \)
- For all \( a \), \((a \lor \neg a)\) and \((a \land \neg a)\) exist and they satisfy \( a \lor \neg a = 1 \) and \( a \land \neg a = 0 \).
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Posets and lattices

Definitions

Orthocomplemented Poset (orthoposet)

$\neg(\neg(a)) = a$

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For all $a$, $(a \lor \neg a)$ and $(a \land \neg a)$ exist and they satisfy $a \lor \neg a = 1$ and $a \land \neg a = 0$. 
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Boolean algebra

Definitions

Distributive lattice

- $x \lor x = x$, $x \land x = x$ (idempotence)
- $x \lor y = y \lor x$, $x \land y = y \land x$ (commutativity)
- $x \lor (y \lor z) = (x \lor y) \lor z$, $x \land (y \land z) = (x \land y) \land z$ (associativity)
- $x \lor (x \land y) = x \land (x \lor y) = x$ (absorption)
- $x \land (y \lor z) = (x \land y) \lor (x \land z)$ (distributivity 1)
- $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ (distributivity 2)

Boolean algebra

An orthocomplemented distributive lattice is a Boolean algebra.
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Distributive lattice

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- \( x \vee y = y \vee x, \ x \wedge y = y \wedge x \) (commutativity)
- \( x \vee (y \vee z) = (x \vee y) \vee z, \ x \wedge (y \wedge z) = (x \wedge y) \wedge z \) (associativity)
- \( x \vee (x \wedge y) = x \wedge (x \vee y) = x \) (absorption)
- \( x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \) (distributivity 1)
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Boolean algebra

An orthocomplemented distributive lattice is a *Boolean algebra*. 
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Kolmogorov Definitions

\[ \mu : \Sigma \rightarrow [0, 1] \]

- satisfying \( \mu(\emptyset) = 0 \)
- \( \mu(A^c) = 1 - \mu(A) \), where \((\ldots)^c\) means set theoretical complement and
- for any denumerable and pairwise disjoint collection \( \{A_i\}_{i \in I} \)
  \[ \mu(\bigcup_{i \in I} A_i) = \sum_i \mu(A_i) \]

Booleanity And Sum Rule
\( \Sigma \) is a boolean algebra and it satisfies
\[ \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \] (inclusion-exclusion principle).
We also have: \( \mu(A \cup B) \leq \mu(A) + \mu(B) \)
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- Cox starts with the boolean algebra of propositions (the propositional calculus of classical logic).
- Studying the symmetries of the boolean algebra, he deduces the general form of the possible measures.
- He deduces Kolmogorovian probability theory with this method (sum rule, inclusion-exclusion principle, product rule, Bayes formulae, etc.).

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Kolmogorov is not the end of the story...

Non-kolmogorovian probability

- Define a map \( s : \mathcal{P}(\mathcal{H}) \to [0; 1] \) such that
  - \( s(0) = 0 \) (0 is the null subspace)
  - \( s(P^\perp) = 1 - s(P) \)
  - for any denumerable and pairwise orthogonal collection of projections \( P_j \), \( s(\sum_j P_j) = \sum_j s(P_j) \).

Quantum Probability

\( s \) is defined on a non-boolean algebra: \( \mathcal{P}(\mathcal{H}) \) is an orthomodular lattice. Gleason: \( \forall s \exists \rho \) such that \( s(P) = \text{tr}(\rho P) \). \( s(a \lor b) \geq s(a) + s(b) \).
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Questions

➤ What happens if the Cox’s method is applied to non-boolean algebras?

➤ Does the underlying logical structure of a given theory determines its probability theory? What happens with information?

➤ What about ontology? Can we provide an ontological framework?


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von Neumann:

“In order to have probability all you need is a concept of all angles, I mean, other than 90. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leaves all angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logic are fixed under some isomorphic transformation, all of probability theory is already fixed... This means however, that one has a formal mechanism in which, logics and probability theory arise simultaneously and are derived simultaneously.”
"It was simultaneous emergence and mutual determination of probability and logic what von Neumann found intriguing and not at all well understood. He very much wanted to have a detailed axiomatic study of this phenomenon because he hoped that it would shed "... a great deal of new light on logics and probability alter the whole formal structure of logics considerably, if one succeeds in deriving this system from first principles, in other words from a suitable set of axioms." He emphasized –and this was his last thought in his address– that it was an entirely open problem whether/how such an axiomatic derivation can be carried out."

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Conclusions
We proceed as follows:

- Let $\mathcal{L}$ be an orthomodular lattice.
- We assume that $\mathcal{L}$ represents the propositional structure of a given system.
- We assume that there is a given state of affairs determined by the preparation of the system.
- We define a function $s : \mathcal{L} \rightarrow \mathbb{R}$ such that $s(a) \geq 0 \forall a \in \mathcal{L}$ and it is order preserving ($a \leq b \rightarrow s(a) \leq s(b)$).
- Under these rather general assumption a probability theory can be developed.
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- We define a function $s : \mathcal{L} \rightarrow \mathbb{R}$ such that $s(a) \geq 0 \forall a \in \mathcal{L}$ and it is order preserving ($a \leq b \rightarrow s(a) \leq s(b)$).
- Under these rather general assumption a probability theory can be developed.
General Method

We proceed as follows:

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It is possible to show that:

- $s(\bigvee \{a_i\}_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} s(a_i)$
- $s(\neg a) = 1 - s(a)$
- $s(0) = 0$

The probability theory defined by these equations is non classical in the general case. If $\mathcal{L}$ is not Boolean, it may happen that $(a \land b) \lor (a \land \neg b) < a$.

and then, (using $(a \land \neg b) \perp (a \land b)$),

$s((a \land \neg b) \lor (a \land b)) = s(a \land \neg b) + s(a \land b) \leq s(a)$. But any Kolmogorovian probability satisfies $s(a) = s(a \land b) + s(a \land \neg b)$. 
**General Method**

**Theory**

- von Neumann
- Lattice of propositions
- Probability theory

**General Framework**

- Quantum probabilities

**General Scheme**

- Group theoretical quantum objects

**Conclusions**

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**Figure**: Schematic diagram of the method.
Outline

Abstract

General Framework
  Lattices
  Posets and lattices
  Boolean algebra

Quantum probabilities
  Kolmogorov
  R. T. Cox
  Quantum Probabilities
  The von Neumann program

General Scheme
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Group theoretical quantum objects

Conclusions
The program posed by von Neumann seems to point in the direction of a special kind of objects...

- **Automorphisms** of the logic seem to play a key role in the program proposed by von Neumann.
- In concrete implementations of quantum observables, symmetry transformations must preserve probability.
- Thus, we have objects such that its propositional lattice is non-boolean, and that probability must remain invariant under these automorphisms.
- Which ontology is compatible with objects of this kind? We look for alternatives.
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Group Theoretical Quantum Ontology

- A quantum object will be an entity having objective properties, but at the same time, each objective property will have associated a symmetry transformation (represented by the action of a group).
- The action of this group will determine in turn a set of non-objective properties.
- To each objective property we can associate an operator. Each element of the orbit defined by this operator will be a phase of the object.

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- Thus, a physical system is a multifaceted structure characterized by a set of invariant objective properties (a kind of dice with (possibly) superposed faces).
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Given that automorphisms of the logic are the elements which allow for the definition of objectivity in the GTQO ontology.

Thus, the notion of GTQO, could provide an ontological framework for the program delineated by von Neumann. This is what we want to explore.

This ontology + the generalization of Cox’s method to non-distributive lattices, could provide a complete realization of the program delineated by von Neumann.
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References