Connecting Abstractions From Hilbert Spaces: From Categories to Frames

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Introduction

The development of Quantum Computation and Information has caused a new wave of studies in Quantum Mechanics. On the practical side we are interested in the implementation of quantum algorithms. On the theoretical side, we seek to develop formal models to increase our understanding of quantum processes.

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We examine two research programs that belong to the second side. They have

- a common goal: crafting a formalism that captures the features of quantum processes
- the same intended application: a formal system capable of proving the correctness of quantum algorithms

Question: can we connect these two research programs ?

Outline

Categorical Quantum Mechanics

The first approach initiated by Abramsky and Coecke, recasts the concepts of Hilbert space Quantum Mechanics in the abstract language of Category Theory. The target of this study is **FdHil**, the category having as objects finite-dimensional Hilbert spaces over the field of complex numbers and as morphisms linear maps.

Categorical Quantum Mechanics

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Theorem (Abramsky and Coecke, [1])

The category **FdHil** *is a dagger compact closed category with biproducts.*

This in particular means that **FdHil** is

1. a symmetric monoidal category
2. a compact closed category
3. a dagger category
4. a category with biproducts

Definition

A *symmetric monoidal category* **C** is a category equipped with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, a distinguished object I and natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$$
$$\sigma_{A,B} : A \otimes B \simeq B \otimes A$$
$$\lambda_A : I \otimes A \simeq A \qquad \rho_A : A \otimes I \simeq A$$

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A *compact closed category* $\mathbf{C}$ is a symmetric monoidal category where to each object A is assigned a dual object A^*, a unit map $\eta_A : I \to A^* \otimes A$ and a counit map $\epsilon_A : A^* \otimes A \to I$ satisfying certain coherence conditions.

Definition

A *dagger category* **C** is a category where to each morphism $f : A \to B$ is associated a morphism $f^\dagger : B \to A$, called the *adjoint* of f, such that

$$Id_A^\dagger = Id_A \qquad (g \circ f)^\dagger = f^\dagger \circ g^\dagger \qquad f^{\dagger\dagger} = f$$

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Definition

Suppose **C** has zero object, products and coproducts. Given objects $A_1, \dots, A_n$, a *biproduct* is an object $A_1 \oplus \dots \oplus A_n$ equipped with morphisms $p_j : A_1 \oplus \dots \oplus A_n \to A_j$ and $q_j : A_j \to A_1 \oplus \dots \oplus A_n$, for $j \in \{1, \dots, n\}$, such that:

- $p_j \circ q_j = Id_{A_j}$
- $p_i \circ q_j = 0_{A_i, A_j}$ for $i \neq j$
- $A_1 \oplus \dots \oplus A_n$ is both the product and the coproduct of $A_1, \dots, A_n$

Dynamic Quantum Logic

The second approach, proposed by Baltag and Smets, exploits the formalism of Propositional Dynamic Logic to design a *Logic of Quantum Programs*, abbreviated in *LQP*. The core ideas behind this logic are two:

1. we can see the states of a physical system as states of a Modal Logic frame
2. the dynamics of the system can be captured by representing measurements as tests and unitary maps as actions

This leads to an abstraction from Hilbert spaces to labelled transition systems.

Quantum dynamic frames

Definition

Given a Hilbert space H, a *quantum dynamic frame* is a tuple $\langle \Sigma_H, \{\xrightarrow{P_a?}\}_{a \in L_H}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$ such that:

1. Σ_H is the set of all one-dimensional linear subspaces of H

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1. Σ_H is the set of all one-dimensional linear subspaces of H
2. $\{\xrightarrow{P_a?}\}_{a \in L_H}$ is a family of *quantum tests*, partial maps from Σ_H into Σ_H associated to the projectors of the Hilbert space H. Given $\overline{v} \in \Sigma_H$, they are defined as $\xrightarrow{P_a?} (\overline{v}) = \overline{P_a(v)}$.

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3. $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ is a collection of partial maps from Σ_H into Σ_H associated to the unitary maps from H into H. As for projectors, their definition is $\xrightarrow{U} (\overline{v}) = \overline{U(v)}$.

LQP

Given a set of atomic propositions *At* and a set of atomic actions *AtAct*, the set of formulas $\mathcal{F}_{LQP}$ and the set of actions *Act* are built by mutual recursion as follows:

$$\psi ::= p \mid \neg\psi \mid \psi \wedge \phi \mid [\pi]\psi$$
$$\pi ::= U \mid \pi^{\dagger} \mid \pi \cup \pi' \mid \pi ; \pi' \mid \psi ?$$

where $p \in At$ and $U \in AtAct$.

LQP^n

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Consider as semantics only the quantum dynamic frames given by n-th tensor products of 2-dimensional Hilbert spaces (systems of n qubits).
Enrich the language with

$$\psi ::= \top_I \mid 1 \mid + \mid p \mid \neg\psi \mid \psi \wedge \phi \mid [\pi]\psi$$

$$\pi ::= triv_I \mid U \mid \pi^\dagger \mid \pi \cup \pi' \mid \pi;\pi' \mid \psi?$$

Theorem (Baltag and Smets, [2])

There is a proof system for LQP^n *which is sound and proves the correctness of some quantum protocols.*

Modal Logics for Small Categories

A bridge between the two research projects can be found only by connecting the two underlying formalisms, Category Theory and Modal Logic. To this end, we describe a way to extract a modal logic frame from a small category and a functor into **Rel**.

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Definition

Given a small category $\mathbf{C}$ and a functor $U : \mathbf{C} \to \mathbf{Rel}$, a $(\mathbf{C}, U)$-frame is a pair $\langle W, Rel \rangle$ such that

- $W := \bigcup\{U(I) | I \in \mathbf{C}_0\}$
- $Rel := \{U(f) | f \in \mathbf{C}_1\}$

Notice that if $\mathbf{C}$ is small then W is the union of set-many sets, and thus is a set. Similarly, as there are set-many morphisms in $\mathbf{C}$, Rel will be a set.

Call Γ the class of all $(\mathbf{C}, U)$-frames.

Modal Logics for $(\mathbf{C}, U)$-frames

Notice that the $(\mathbf{C}, U)$-frames are labelled transition systems, or in philosophical terminology, sets of worlds equipped with accessibility relations.

We can therefore design modal logics for classes of $(\mathbf{C}, U)$-frames. For example we can use the language

$$\psi ::= p \mid \neg\psi \mid \psi \wedge \phi \mid [R]\psi$$

for R relation in Rel, or the basic modal language

$$\psi ::= p \mid \neg\psi \mid \psi \wedge \phi \mid \Box\psi$$

Examples of *LSC* Logics

We call Logic for Small Categories, *LSC*, a modal logic for a subclass of Γ. We mention briefly some interesting *LSC* logics which are not related to QM:

- *DLT*, a dynamic logic with types designed to describe typed processes, whose proof system is *sound* with respect to Γ; the proof system is *complete* when we restrict to finitely many types.
- *S*4 is *sound and complete* with respect to Γ, when the satisfaction of the diamond operator is adapted to the new setting.
- When we restrict our attention to a certain kind of functors, called singleton functors, we get a class of modal logic frames whose logic contains the validities of the Hybrid Logic of *S*4.

Modal Logics for Locally Small Categories

Now we want to study the application of this procedure to locally small categories, in order to apply it to **FdHil**. This can be done replacing the idea

one category, one frame

with the slogan

one category, *many* frames

More precisely, given a locally small category **C**, we can consider the class of modal logic frames generated by all the small subcategories of **C**.

Logics for **FdHil**

We now apply our procedure to obtain a class of modal logic frames from **FdHil**. In this case we are interested in having one frame for each physical system. Thus we will consider the frames generated by the subcategories of **FdHil** with only one object.

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Notice that the procedure depends on the choice of a functor **FdHil** $\rightarrow$ **Rel**. Each functor specifies a different *abstraction* from Hilbert spaces. Each functor produces a different class of modal logic frames, and therefore a different modal logic.

Logics for **H** and S

Consider the functor $S : \mathbf{FdHil} \to \mathbf{Rel}$ defined as

$$H \mapsto \Sigma_H$$
$$L : H \to V \mapsto S(L) : \Sigma_H \to \Sigma_V$$

where Σ_H is the set of one-dimensional closed linear subspaces of H and the functions $S(L)$ are the partial functions defined as $S(L)(\overline{v}) = \overline{L(v)}$.

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For **H** full subcategory of **FdHil** containing only the Hilbert space H, an $(\mathbf{H}, S)$-frame is a pair $\langle W, Rel \rangle$ defined as

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$(\mathbf{H}, S)$-frames "contain" quantum dynamic frames, hence:

Corollary

The logic of the class of $(\mathbf{H}, S)$-frames in the language $\mathcal{F}_{LQP}$ contains all the theorems of LQP. Similarly for LQP^n, when the class of frames is suitably restricted.

Logics for **H** and F

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Notice that the modal logic frames given by S do not contain any probabilistic information. We now define a functor that will give us a richer semantics. Consider F : **FdHil** $\rightarrow$ **Rel** defined as:

$$H \mapsto A_H$$
$$L : H \rightarrow V \mapsto F(L) : A_H \rightarrow A_V$$

The set A_H is the set of functions $s_\rho : L_H \rightarrow [0, 1]$, where L_H is the lattice of closed linear subspaces of H, defined as

$$s_\rho(a) = tr(P_a \rho)$$

where P_a is the projector associated to the subspace a and ρ is a density operator on H. A linear map $L : H \rightarrow V$ is sent to the partial function $F(L) : A_H \rightarrow A_V$ where

$$F(L)(s_\rho) = s_{\frac{L\rho L^\dagger}{tr(L\rho L^\dagger)}}$$

Logics for **H** and F

A $(\mathbf{H}, F)$-frame is thus pair $\langle W, Rel \rangle$ such that

- $W := A_H$
- $Rel := \{F(L) | L : H \to H\}$

These frames contain *all mixed states* and encode the *probabilistic information* about the outcome of measurements. We can design a language that captures these additional aspects.

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Lemma

There is a natural transformation $\delta : S \to F$.

Thanks to this lemma we can turn every $(\mathbf{H}, F)$-frame into a $(\mathbf{H}, S)$-frame.

Theorem

Upon translation, all the theorems of LQP (LQP^n) are validities of the class of $(\mathbf{H}, F)$-frames (given by compound systems) in the new language.

Conclusions and future work

We have seen that, choosing the appropriate functor, we can extract the semantics of LQP from **FdHil**. This constitutes the formal link between the two approaches that we surveyed.
The procedure also suggested a way to enrich the semantics of LQP. We designed a logic for such semantics and proved that it preserves all the correctness results of LQP^n.

In future work we will:

- craft a proof system for our new logic and try to prove the correctness of quantum protocols where probability plays an essential role;
- explore the connections with other logics in the area
- investigate whether some of the categorical structure of **FdHil** can be transferred to the image of F
- study the relations with other abstractions from Hilbert spaces

And of course, there is the general issue of the interplay between Category Theory and Modal Logic...

References

S. Abramsky and B. Coecke.
Categorical quantum mechanics.
Handbook of Quantum Logic Vol. II, 2008.

A. Baltag and S. Smets.
LQP, The Dynamic Logic of Quantum Information.
Mathematical Structures in Computer Science, 16(3):491–525, 2005.

G. Cinà.
On the connection between the categorical and the modal logic approaches to Quantum Mechanics.
Master thesis, ILLC, 2013.