Being serious about permutation invariance in quantum mechanics

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Précis

1. Permutation invariance ought to be construed as an "analytic symmetry", i.e. an artefact of representational redundancies. I call this doctrine **anti-factorism**.
2. Anti-factorism has important and overlooked consequences:
    1. The full state space accessible to an assembly of elementary systems does not have a straightforward **tensor product structure**.
    2. The **partial trace** operation cannot be taken to yield information about the state of a constituent system.
    3. Non-separability provides an inadequate (because too weak) criterion for **entanglement**.
3. In this talk I will advocate:
    1. A decomposition of (parts of) the assembly's state space into state spaces for the proper constituents. These constituents ought to be construed as **absolutely discernible**.
    2. An alternative to the partial trace operation for extracting states of constituent systems, which vindicates the above claim.
    3. An alternative criterion (and continuous measure) for entanglement (according to which e.g. fermions are *not* always entangled).

Outline

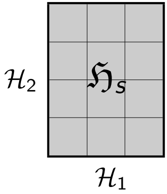
Distinguishable systems (1)

An assembly of two "distinguishable" systems:

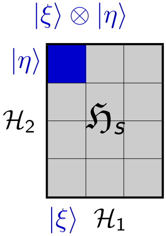
- $\mathcal{H}_1$ – Hilbert space of system 1
- $\mathcal{H}_2$ – Hilbert space of system 2
- $\mathcal{H}_1 \otimes \mathcal{H}_2 =: \mathfrak{H}_s$ – Hilbert space of assembly
- $\mathcal{H}_1$ *as it occurs as a factor of* $\mathfrak{H}_s$ continues to represent the possible states for system 1, and $\mathcal{H}_2$ to represent the possible states for system 2.

- So, e.g., $|\xi\rangle \otimes |\eta\rangle$ is interpreted as: system 1 being in the state (represented by) $|\xi\rangle$ and system 2 being in the state (represented by) $|\eta\rangle$.
- Any non-separable state counts as **entangled**.

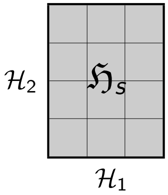
Distinguishable systems (2)



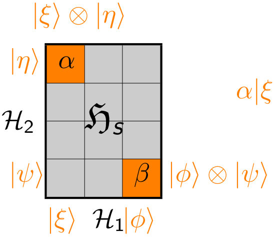
Distinguishable systems (2)



Distinguishable systems (2)



Distinguishable systems (2)



$$\alpha|\xi\rangle \otimes |\eta\rangle + \beta|\phi\rangle \otimes |\psi\rangle$$

Indistinguishable systems

An assembly of two "indistinguishable" systems:

- $\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathcal{H}$ – single system Hilbert space
- $\mathcal{H} \otimes \mathcal{H}$ – *prima facie* assembly Hilbert space.

- Impose **permutation invariance**:

$$\forall \pi \in S_N \quad : \quad \langle \psi | P_\pi^\dagger Q P_\pi | \psi \rangle = \langle \psi | Q | \psi \rangle \tag{1}$$

- Induced superselection, with superselection sectors

$$\mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \quad \text{and} \quad \mathcal{A}(\mathcal{H} \otimes \mathcal{H})$$

Factorism defined

$$\mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \quad \text{or} \quad \mathcal{A}(\mathcal{H} \otimes \mathcal{H})$$

Factorism:
(As in the "distinguishable" case) each factor Hilbert space $\mathcal{H}$ represents the possible states for one of the systems.

- One may take each factor Hilbert space label as a particle label.
- Again, any non-separable state, e.g. $\frac{1}{\sqrt{2}}(|\xi\rangle \otimes |\eta\rangle + |\eta\rangle \otimes |\xi\rangle)$ is interpreted as **entangled**.

Decomposing the joint Hilbert space properly

- Factorism does not decompose the right Hilbert space.
  It decomposes $\mathcal{H} \otimes \mathcal{H}$; it should decompose $\mathcal{S}(\mathcal{H} \otimes \mathcal{H})$.
- We need to think about how to decompose the right Hilbert space (or its associated algebra).

Anti-factorist Credo: *Factor Hilbert space labels represent nothing.*

Outline

Natural decompositions

- Following Zanardi (2001) and Zanardi, Lidar & Lloyd (2004), a natural decomposition requires finding a **tensor product structure**.
- So we seek two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ such that $\mathcal{S}(\mathcal{H} \otimes \mathcal{H})) \cong \mathcal{H}_1 \otimes \mathcal{H}_2$.
- But $\dim(\mathcal{S}(\mathcal{H} \otimes \mathcal{H}))$ might be prime!
- So instead seek natural decompositions of **subspaces** of the joint Hilbert space $\mathcal{S}(\mathcal{H} \otimes \mathcal{H})$:

$$\mathcal{S}(\mathcal{H} \otimes \mathcal{H}) = \bigoplus_i \mathfrak{S}_i, \quad \text{where each } \mathfrak{S}_i \cong \mathcal{H}_1^{(i)} \otimes \mathcal{H}_2^{(i)} \tag{2}$$

- Particle assemblies as like political assemblies, which survive their individual members.

Qualitative Individuation (1)

- These subspaces are to be picked out by **qualitative individation**.
- ‘Qualitative individuation’ means to pick out by appeal to qualitative properties and relations.

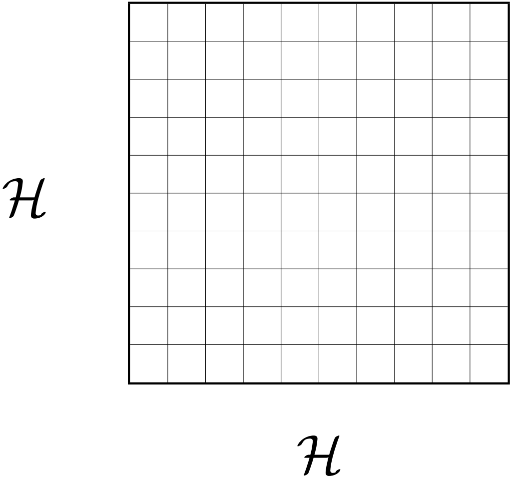
- Projectors E_α, E_β on $\mathcal{H}$ – **individuation criteria**
- *We must have* $E_\alpha \perp E_\beta$; i.e. $E_\alpha E_\beta = E_\beta E_\alpha = 0$.
- *We may have* $\dim(E_\alpha)$, $\dim(E_\beta) > 1$ and $E_\alpha + E_\beta < \mathbb{1}$.
- e.g.: $E_\alpha = \int_{-\infty}^{0} \mathrm{d}E_x \otimes \mathbb{1}_{\text{spin}}$; $E_\beta = \int_{0}^{\infty} \mathrm{d}E_x \otimes \mathbb{1}_{\text{spin}}$

- Then the projector on $\mathcal{H} \otimes \mathcal{H}$:

$$\mathcal{E} := E_\alpha \otimes E_\beta + E_\beta \otimes E_\alpha \tag{3}$$

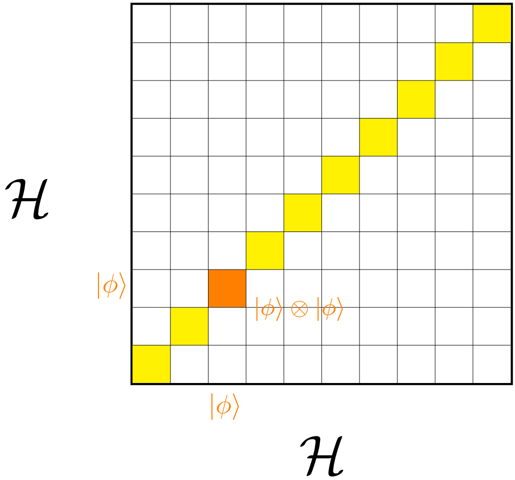
picks out a subspace, aka an **individuation block**.

Qualitative individuation (2)



Each square
represents
a product state

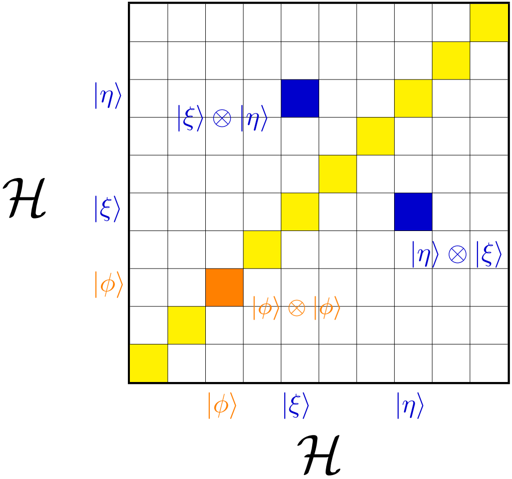
Qualitative individuation (2)



Each square represents a product state

Condensed states along diagonal

Qualitative individuation (2)

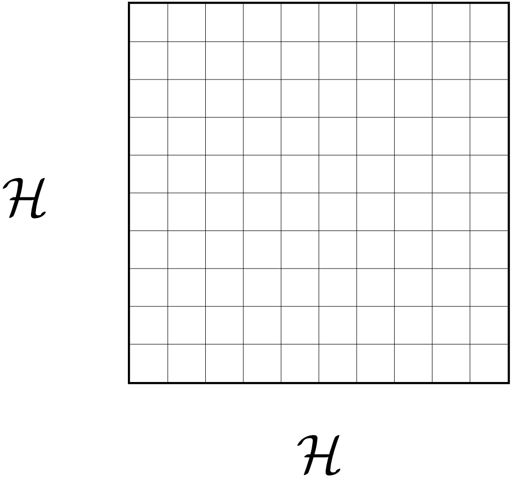


Each square represents a product state

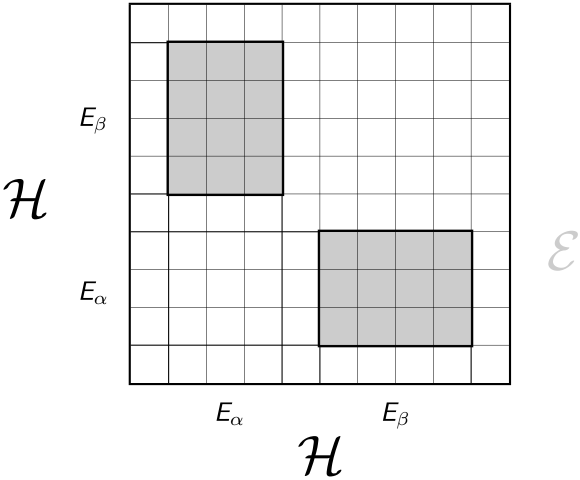
Condensed states along diagonal

Permutes reflected across diagonal

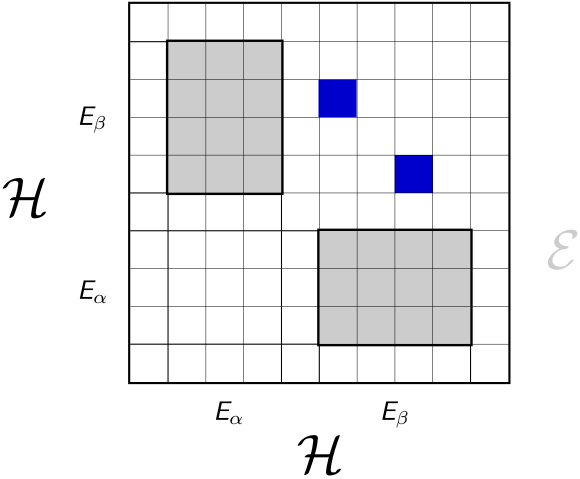
Qualitative individuation (2)



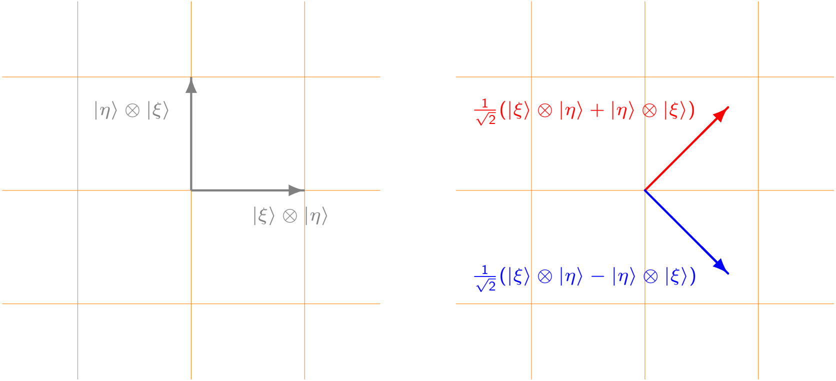
Qualitative individuation (2)



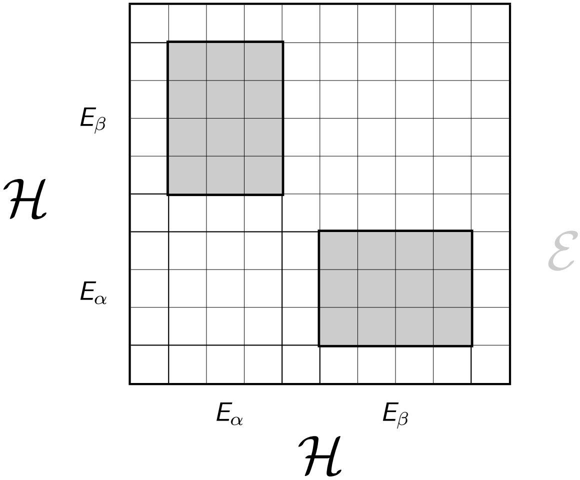
Qualitative individuation (2)



Qualitative individuation (2)

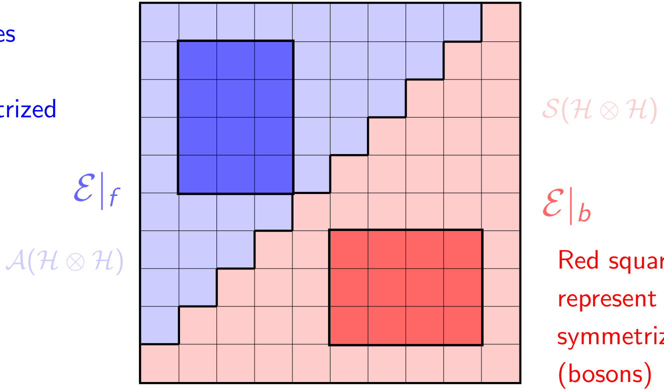


Qualitative individuation (2)



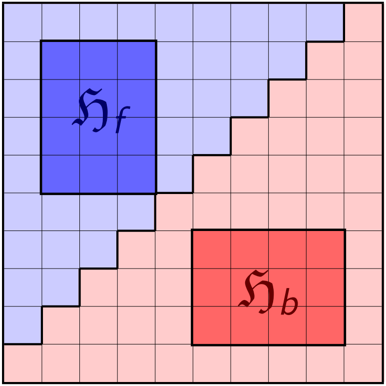
Qualitative individuation (2)

Blue squares represent antisymmetrized states (fermions)



Red squares represent symmetrized states (bosons)

Qualitative individuation (2)



Outline

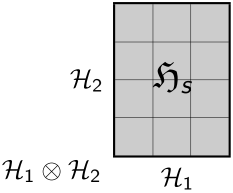
Reps and representation

A **rep** of a $*$-algebra $\mathfrak{A}$ is an ordered pair $\langle \mathfrak{H}, \pi \rangle$ of a Hilbert space $\mathfrak{H}$ and a $*$-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$.

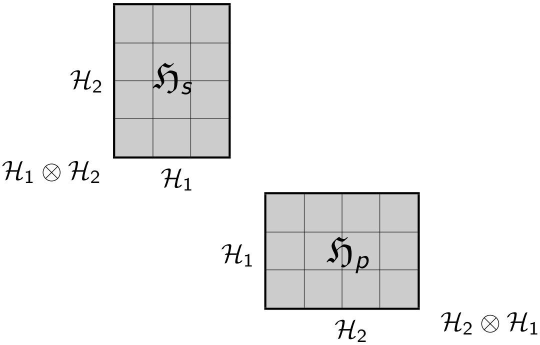
Two reps $\langle \mathfrak{H}, \pi \rangle$ and $\langle \mathfrak{K}, \phi \rangle$ are **unitarily equivalent** iff there is a unitary $U : \mathfrak{H} \to \mathfrak{K}$ such that $U\pi[\mathfrak{A}] = \phi[\mathfrak{A}]U$.

> If two reps are unitarily equivalent, and any physical quantity not represented is invariant between them, then the two reps are equally good representations of any given physical system.

An example of unitary equivalence



An example of unitary equivalence



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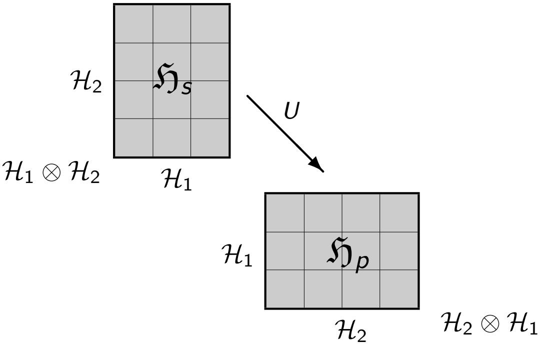
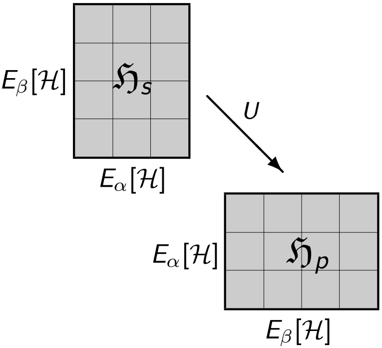
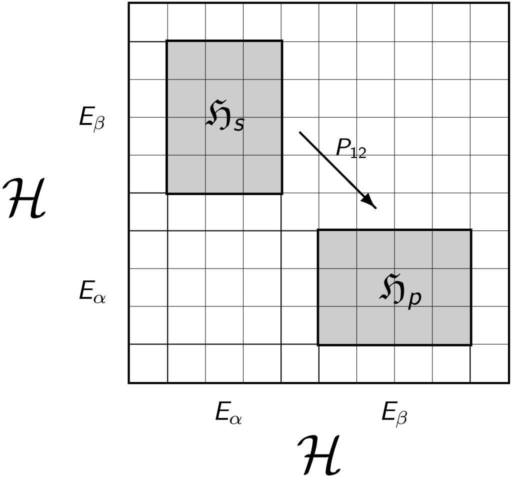


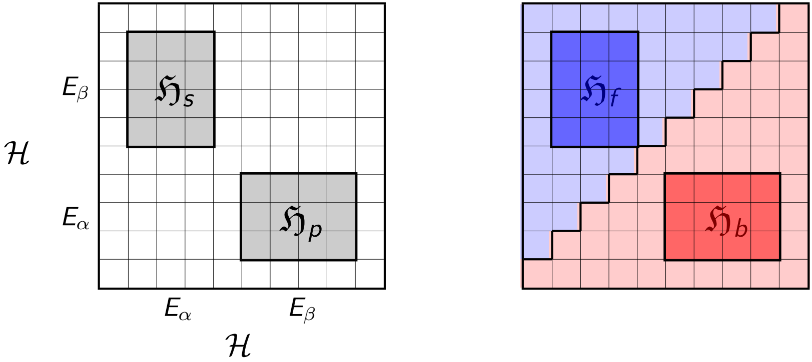
Qualitative Individuation, contd.



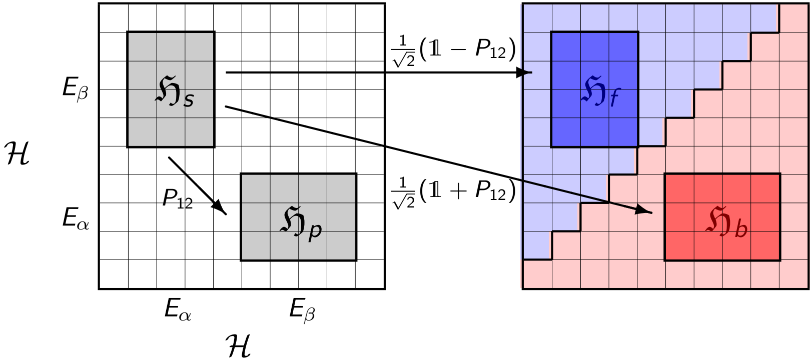
Qualitative Individuation, contd.



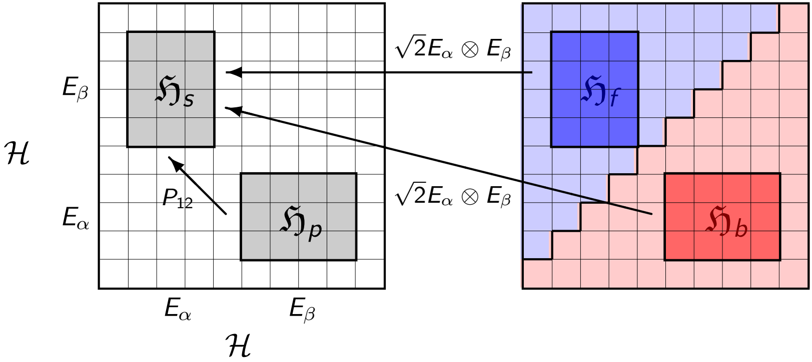
Qualitative Individuation, contd.



Qualitative Individuation, contd.



Qualitative Individuation, contd.



Qualitative individuation, contd.

- $\mathfrak{H}_s$, $\mathfrak{H}_p$, $\mathfrak{H}_f$ and $\mathfrak{H}_b$ are all unitarily equivalent:

$$\mathfrak{H}_b \cong \mathfrak{H}_f \cong \mathfrak{H}_p \cong \mathfrak{H}_s = \mathcal{H}_1 \otimes \mathcal{H}_2$$

- So we have found a tensor product structure for $\mathfrak{H}_b$ and $\mathfrak{H}_f$.
- Also, all relevant physical quantities are represented in these Hilbert spaces (assuming the anti-factorist credo).
- So we should give the *same* physical interpretation to all four Hilbert spaces:
  - One system has a state in $\text{ran}(E_\alpha)$, and the other has a state in $\text{ran}(E_\beta)$.
  - But $E_\alpha \perp E_\beta$: so the two systems possess different properties—i.e. they are **absolutely discernible**.
- All this holds so long as the assembly's state vector lies in the **individuation block** determined by E_α & E_β.

Outline

Constituent systems considered alone

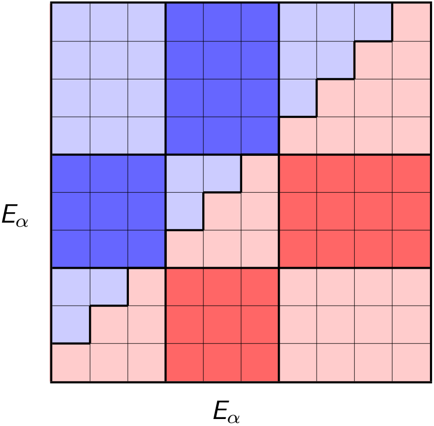
How do we calculate expectation values for a *single* constituent system?

Two proposals:

1. **Individuation:** demand *unique* instantiation of some individuation criterion E_α.
2. **Averaging:** average over all the constituent systems that satisfy E_α.

It's surely best to treat these as complementary tools, rather than rivals.

Lonely systems: individuation

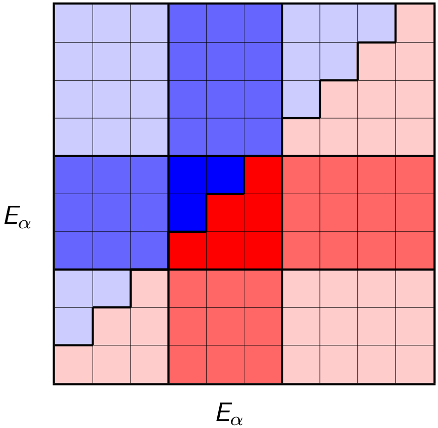


$$\mathcal{E}_\alpha := E_\alpha \otimes E_\alpha^\perp + E_\alpha^\perp \otimes E_\alpha$$
$$\iota_\alpha(A) := E_\alpha A E_\alpha \otimes E_\alpha^\perp + E_\alpha^\perp \otimes E_\alpha A E_\alpha$$

$$\langle A \rangle_\alpha := \frac{\langle \iota_\alpha(A) \rangle}{\langle \mathcal{E}_\alpha \rangle} = \frac{\langle \iota_\alpha(A) \rangle}{\langle \iota_\alpha(\mathbb{1}) \rangle}$$

- Russell vs. Strawson?
- $\iota_\alpha[\mathcal{B}(\mathcal{H})] \cong \mathcal{B}(\mathrm{ran}(E_\alpha))$
- A special case of multi-particle decomposition only for $N = 2$

Lonely systems: averaging



$$n_\alpha := E_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes E_\alpha$$
$$\pi_\alpha(A) := E_\alpha A E_\alpha \otimes \mathbb{1} + \mathbb{1} \otimes E_\alpha A E_\alpha$$

$$\langle A \rangle_\alpha := \frac{\langle \pi_\alpha(A) \rangle}{\langle n_\alpha \rangle} = \frac{\langle \pi_\alpha(A) \rangle}{\langle \pi_\alpha(\mathbb{1}) \rangle}$$

- Contributions from "doubly occupied" states are counted twice
- $\pi_\alpha[\mathcal{B}(\mathcal{H})] \ncong \mathcal{B}(\mathrm{ran}(E_\alpha))$

Outline

Reduced density operators for QI'd systems

We know from Gleason's theorem that some density operator ρ_α exists such that

$$\forall A \in \mathcal{B}(\mathcal{H}) : \quad \langle A \rangle_\alpha = \mathrm{Tr}(\rho_\alpha A) \tag{4}$$

The specific form (in orthobasis $\{|\phi_i\rangle\}$) is

$$\rho_\alpha = \frac{1}{\langle n_\alpha \rangle} \sum_{i,j} \mathrm{Tr}\left[\rho \ \iota_\alpha \left(|\phi_j\rangle\langle\phi_i|\right)\right] |\phi_i\rangle\langle\phi_j| \tag{5}$$

Special cases:

- $E_\alpha = |\alpha\rangle\langle\alpha|$. Then $\rho_\alpha = |\alpha\rangle\langle\alpha|$, so long as $\langle n_\alpha \rangle > 0$.
- $E_\alpha = \mathbb{1}$. Then

$$\rho_\alpha = \frac{1}{N} \sum_{k=1}^{N} \sum_{i,j} \mathrm{Tr}\left[\rho \ \left(|\phi_j\rangle\langle\phi_i|\right)_k\right] |\phi_i\rangle\langle\phi_j| = \frac{1}{N} \sum_{k=1}^{N} \rho_k = \overline{\rho} \tag{6}$$

So for bosons & fermions, all the ρ_k—calculated using the partial trace—give the (same) state of the "average system".

Outline

Ubiquitous and unique systems

For any state $|\psi\rangle$ of the assembly, call a constituent system, QI'd with some IC E_α, **ubiquitous and unique** in $|\psi\rangle$ iff E_α is instantiated *exactly once* in every separable term of $|\psi\rangle$:

$$\text{System } \alpha \text{ is u \& u} \quad \text{iff} \quad (E_\alpha \otimes E_\alpha^\perp + E_\alpha^\perp \otimes E_\alpha)|\psi\rangle = |\psi\rangle$$

- ICs whose associated systems are u & u are anti-factorist surrogates for **particle labels**.
- If an N-system state has support in some individuation block, then $\exists$ N orthogonal ICs for systems which are u & u.
- U & u systems "know" about the whole state, and we don't get in trouble with multiple occupation.
- This is invaluable in providing a new definition of entanglement...

Outline

The need for a better notion

- **Strategy:** Give the same interpretation to each individuation block as is given to its u.e. product spaces.

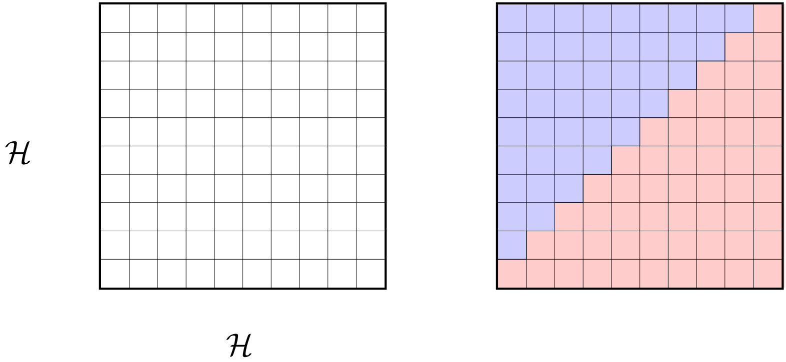
- Take any state which maps from a product state under U. It will be of the form

$$\frac{1}{\sqrt{2}}(|\xi\rangle \otimes |\eta\rangle \pm |\eta\rangle \otimes |\xi\rangle) \tag{7}$$

- Since the corresponding product state is non-entangled (separable), we ought also to count (7) as non-entangled.

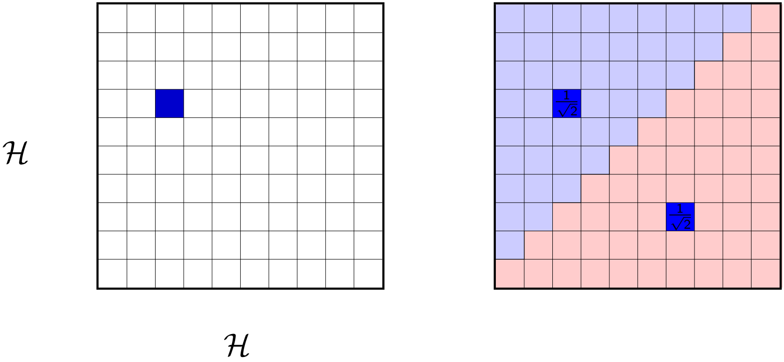
- cf. Ghirardi, Marinatto & Weber (2002), Ghirardi & Marinatto (2003, 2004, 2005) . . .

Entanglements compared



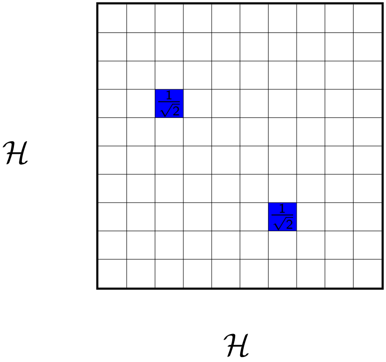
Entanglements compared

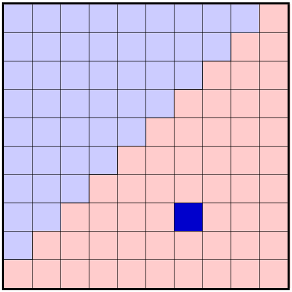
non-entanglement



Entanglements compared

non-GMW-entanglement (bosons)



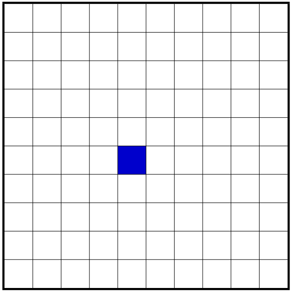


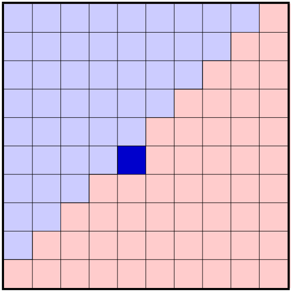
Entanglements compared

non-entanglement

non-GMW-entanglement (bosons)

$\mathcal{H}$

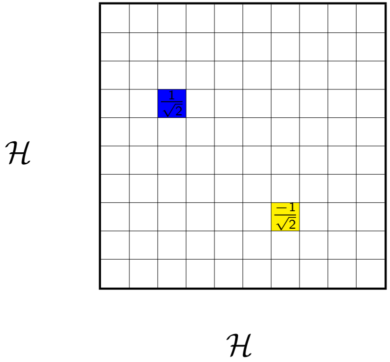


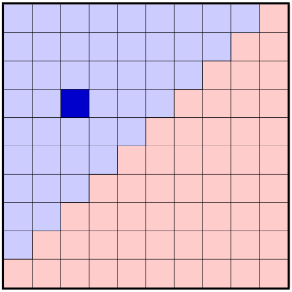


$\mathcal{H}$

Entanglements compared

non-GMW-entanglement (fermions)





Outline

GMW-entanglement

A bosonic (fermionic) state is **GMW-entangled** iff it is *not* the (anti-) symmetrization of a separable state.

- Nice result 1:
  - An assembly's state is non-entangled iff its constituent systems occupy **pure** states.
  - An assembly's state is non-GMW-entangled iff **1D** ICs suffice to individuate the constituent systems.
- Nice result 2: For two systems:
  - An assembly's state is entangled iff it violates a Bell inequality for some beables (Gisin 1991).
  - An assembly's state is GMW-entangled iff it violates a Bell inequality for some *symmetric* beables.

Outline

A continuous measure of GMW-entanglement

In **eligible** N-system states, we may define a continuous measure of GMW-entanglement. Eligible := there are N u & u systems.

Take the von Neumann entropy of the reduced density operator of (any of) the u & u systems.

$$\begin{aligned} E_{GMW}(|\psi\rangle) &= S(\rho_\alpha) := -\text{Tr}(\rho_\alpha \log_2 \rho_\alpha) \\ &= -\text{Tr}[\overline{\rho} \log_2(N\overline{\rho})] \\ &= S(\overline{\rho}) - \log_2 N \end{aligned}$$

- We use $N\overline{\rho} = \text{diag}(\rho_\alpha, \rho_\beta, \ldots)$.
- $S(\overline{\rho})$ is independent of the ICs $E_\alpha, E_\beta, \ldots$, so E_{GMW} is too.
- For eligible states, $E_{GMW}(|\psi\rangle) = 0$ iff $|\psi\rangle$ is not GMW-entangled.
- $E_{GMW} \leqslant \log_2(\frac{d}{N})$. For fermions, $E_{GMW} \leqslant \log_2(\llcorner \frac{d}{N} \lrcorner)$.

Outline

Fermions are always absolutely discernible (1)

Constituent systems are individuated—and all the foregoing is valid—*only so long as* the assembly's state has support in some individuation block.

Can an individuation block always be found for the assembly's state?

- Bosons: **No:** non-zero amplitudes for multiply-occupied states!
- Fermions: **Yes:** by using Slater decompositions...

  (See Schliemann, Cirac, Kuś, Lewenstein & Loss (2001); Eckert, Schliemann, Bruß & Lewenstein (2002).)

Fermions are always absolutely discernible (2)

4	c_{14}	c_{24}	c_{34}	
3	c_{13}	c_{23}		$-c_{34}$
2	c_{12}		$-c_{23}$	$-c_{24}$
1		$-c_{12}$	$-c_{13}$	$-c_{14}$
	1	2	3	4

Fermions are always absolutely discernible (2)

We now implement the following theorem (see e.g. Mehta 1977)...

For any complex anti-symmetric $n \times n$ matrix M, there is a unitary transformation U such that $\tilde{M} := UMU^T$ has the form

$$\tilde{M} = \mathrm{diag}[Z_1, \ldots Z_r, Z_0]$$

$$\text{where} \quad Z_k := iz_k\sigma_2 \quad \text{and} \quad Z_0 := \mathbb{O}^{(n-2r)}$$

Fermions are always absolutely discernible (2)

	$1'$	$2'$	$3'$	$4'$
$4'$			$d_{3'4'}$	
$3'$				$-d_{3'4'}$
$2'$	$d_{1'2'}$			
$1'$		$-d_{1'2'}$		

Fermions are always absolutely discernible (2)

	$1'$	$3'$	$2'$	$4'$
$4'$		$d_{3'4'}$		
$3'$				$-d_{3'4'}$
$2'$	$d_{1'2'}$			
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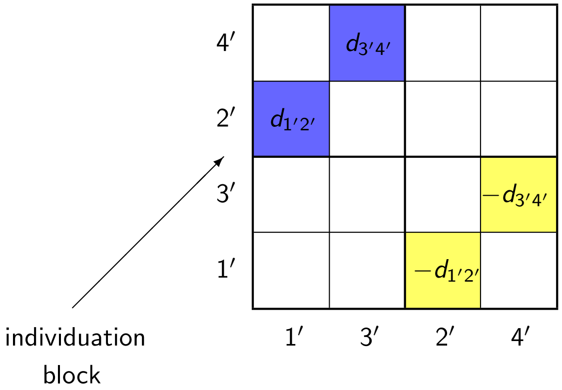
Fermions are always absolutely discernible (2)



Trans-branch identity (1)

But the basis states $1', 2', 3', 4'$ may be grouped together differently. . .

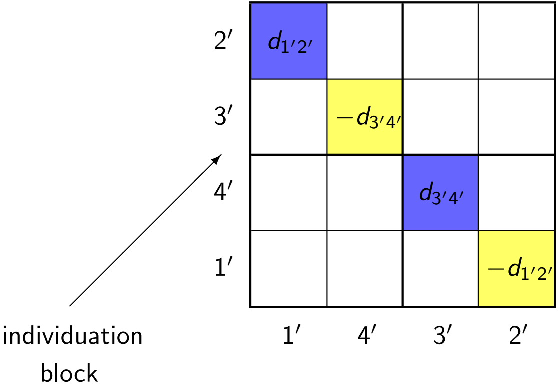
Trans-branch identity (2)

$4'$			$d_{3'4'}$	
$3'$				$-d_{3'4'}$
$2'$	$d_{1'2'}$			
$1'$		$-d_{1'2'}$		
	$1'$	$2'$	$3'$	$4'$

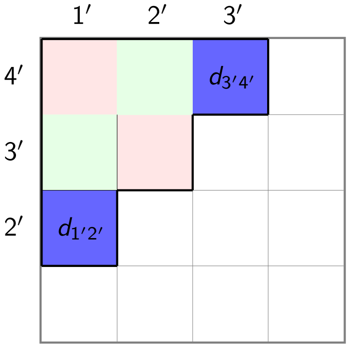
Trans-branch identity (2)

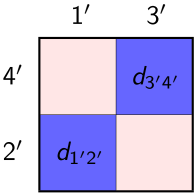
$4'$			$d_{3'4'}$	
$3'$		$-d_{3'4'}$		
$2'$	$d_{1'2'}$			
$1'$				$-d_{1'2'}$
	$1'$	$4'$	$3'$	$2'$

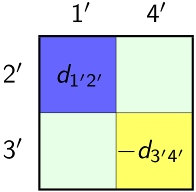
Trans-branch identity (2)



Trans-branch identity (3)







No uniquely natural
"trans-branch identity"
for u & u systems!

Outline

Rival natural decompositions

Any non-GMW-entangled fermionic state can be naturally decomposed in a *variety* of bases.

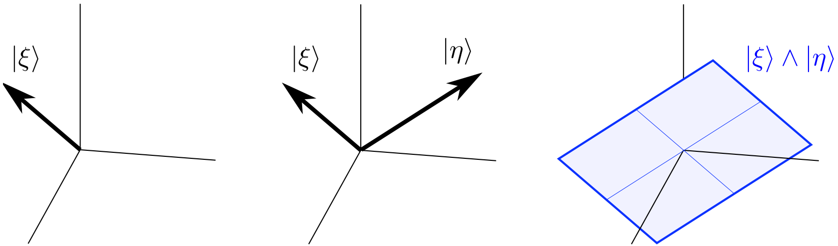
The most well-known example if the spherically symmetric state

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\left(|\leftarrow\rangle \otimes |\rightarrow\rangle - |\rightarrow\rangle \otimes |\leftarrow\rangle\right) \tag{8}$$

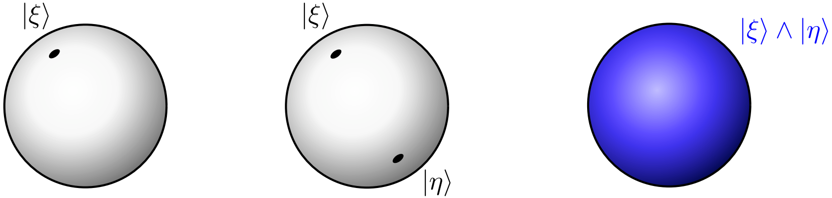
But the result is completely general, applying to all non-GMW fermionic states.

This has to do with the fact that fermionic states are **wedge products** of single-system states.

Non-entangled states represented in $\mathcal{H}$



1 particle | 2 dist. particles | 2 fermions



Outline

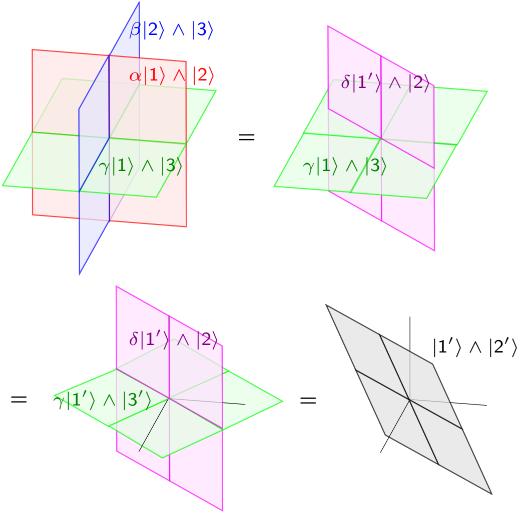
Entanglement is hard to find

Recall that for fermionic states $E_{GMW} \leqslant \log_2(\lfloor \frac{d}{N} \rfloor)$.

This is because single-system states need to be used both for *individuating* the constituent systems (i.e. ensuring that they are u & u) and for generating the variety required for *entanglement*.

Imagine distributing d single-system states between N systems. The degree of entanglement available is constrained by the system with the *least* number of states allocated to it.

No entanglement in $\mathbb{C}^3$



$$|1'\rangle := \frac{\alpha}{\delta}|1\rangle - \frac{\beta}{\delta}|3\rangle$$

$$\delta := \sqrt{|\alpha|^2 + |\beta|^2}$$

$$= \sqrt{1 - |\gamma|^2}$$

$$|3'\rangle := \frac{\beta^*}{\delta}|1\rangle + \frac{\alpha^*}{\delta}|3\rangle$$

$$|2'\rangle := \delta|2\rangle + \gamma|3'\rangle$$

Summary

1. Taking permutation invariance *seriously* means taking factor Hilbert space labels to represent *nothing*.
2. The Hilbert space of an assembly under PI is not *globally* naturally decomposable, but "off-diagonal" subspaces of it are.
3. States for constituent systems *cannot* be obtained by the usual partial trace. They must be *qualitatively individuated*.
4. The right notion of entanglement under PI is *GMW-entanglement* $\neq$ non-separability.
5. Any non-GMW-entangled state of N fermions is aptly represented by an N-dimensional subspace of the *single-system* Hilbert space.